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MECHANICS

Dynamic Compression of a Cold Gas Sphere

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Abstract—In this study, an exact solution of the problem on the dynamic compression of a cold gas sphere of finite size is constructed. The parameters at the shock-wave front and at the gas sphere boundary are mutually independent. The conditions of separation of variables and the initial differential equations in partial derivatives are formulated at a specified shock-wave trajectory for two ODE systems, one of which contains quantities that depend on time alone, and the second one contains quantities that depend on the dimensionless variable alone. The exponent in the dimensionless variable is determined from the condition of an absence of strong discontinuities in the gas flow between the shock wave and gas sphere boundary.

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Attempts to construct analytical solutions in order to create the conditions for energy cumulation have been pursued for more than half a century. The solutions were usually constructed for an infinite region. As for the technical facilities, they always have finite sizes. In contrast to [1–5], let us consider the dynamic compression of a finite-size gas sphere in the following statement. A cold ideal gas with parameters $\rho_0 = \text{const}$, $P_0 = 0$, and $u_0 = 0$ (ρ is the density, P is the pressure, and u is the velocity) is arranged at instant $t = t_0$ in a spherical region with radius $r = r_0$. Boundary velocity $u_{10} < 0$ is specified at point $t = t_0$, $r = r_0$. If $t > t_0$, a shock wave (SW) will spread from this point into the gas with velocity $D_1 < 0$; at the SW front,

$$\rho_1 = \frac{\gamma+1}{\gamma-1}\rho_0, \quad u_1 = \frac{2}{\gamma+1}D_1, \quad P_1 = \frac{2}{\gamma+1}\rho_0 D_1^2. \tag{1}$$

The gas flow between the SW and the sphere boundary (SB) is determined by the equations

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{2\rho u}{r} = 0,$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial P}{\partial r} = 0, \tag{2}$$

$$\frac{\partial P}{\partial t} + u\frac{\partial P}{\partial r} + \gamma P\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0. \tag{3}$$

Let us come to new variables t, $\xi(r, t)$ and seek the solution of set of equations (2), (3) in the form

$$P = \alpha_P(t)\Pi(\xi), \quad \rho = \alpha_\rho(t)\delta(\xi),$$
$$u = \alpha_u(t)M(\xi). \tag{4}$$

For dimensionality reasons, we will consider that $\alpha_P = \alpha_\rho\alpha_u^2$. We well denote derivatives with respect to t with a dot above the magnitude and derivatives with respect to ξ by a prime. After the passage to new variables, Eqs. (2), (3) take the form

$$\varphi_1\delta + \omega\xi\delta' + M\delta' + \delta M' + \frac{2M\delta}{\xi} = 0,$$
$$\varphi_2\delta M + \omega\delta\xi M' + \delta MM' + \Pi' = 0, \tag{5}$$

$$\varphi_3\Pi + \omega\xi\Pi' + M\Pi' + \gamma\Pi M' + \frac{2\gamma M\Pi}{\xi} = 0, \tag{6}$$

where

$$\varphi_1 = \frac{\dot\alpha_\rho}{\alpha_\rho\beta}, \quad \varphi_2 = \frac{\dot\alpha_u}{\alpha_u\beta}, \quad \varphi_3 = \frac{\dot\alpha_P}{\alpha_P\beta},$$
$$\omega = \frac{\partial\xi}{\partial t}\frac{1}{\xi\beta}, \quad \beta = \alpha_u\frac{\partial\xi}{\partial r}. \tag{7}$$

Let us divide Eqs. (5) and (6) into two sets, one of which contains quantities depending on t, and the other contains those depending on ξ. For this purpose, it is desirable that

$$\varphi_1(t) = \text{const}, \quad \varphi_2(t) = \text{const},$$
$$\varphi_3(t) = \text{const}, \quad \omega(t) = \text{const}. \tag{8}$$

Conditions (8) are fulfilled if function $\xi(t, r)$ depends linearly on r:

$$\xi = rf(t). \tag{9}$$

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Table

No.	t	U	No.	t	U	No.	t	U
1	0.04	−1.008792	11	0.24	−1.042656	21	0.34	−1.042843
2	0.07	−1.015162	12	0.25	−1.043463	22	0.35	−1.041620
3	0.10	−1.021257	13	0.26	−1.044130	23	0.36	−1.040097
4	0.13	−1.026982	14	0.27	−1.044643	24	0.37	−1.038252
5	0.16	−1.032224	15	0.28	−1.044992	25	0.38	−1.036058
6	0.18	−1.035381	16	0.29	−1.045165	26	0.39	−1.033458
7	0.20	−1.038211	17	0.30	−1.045146	27	0.40	−1.030514
8	0.21	−1.039486	18	0.31	−1.044921	28	0.42	−1.023220
9	0.22	−1.040657	19	0.32	−1.044474	29	0.45	−1.008354
10	0.23	−1.041717	20	0.33	−1.043787	30	0.50	−0.9699473

If we specify the SW trajectory $r_1 = r_0\left(\frac{t_f - t}{t_f - t_0}\right)^n$, where t_f is the SW focusing instant, and accept $\xi_1 = 1$ for the SW, then we will derive expressions for $f(t)$ and $\xi(r,t)$ from (9):

$$f(t) = \frac{1}{r_0}\left(\frac{t_f - t_0}{t_f - t}\right)^n, \quad \xi = \frac{r}{r_0}\left(\frac{t_f - t_0}{t_f - t}\right)^n. \qquad (10)$$

Differentiating $r_1(t)$ with respect to t, we will find the SW velocity

$$D_1 = -\frac{r_0 n}{t_f - t_0}\left(\frac{t_f - t}{t_f - t_0}\right)^{n-1}. \qquad (11)$$

Two equations at the SW front follow from (1), (4), and (11):

$$u_1 = -\frac{2}{\gamma+1}\frac{r_0 n}{t_f - t_0}\left(\frac{t_f - t}{t_f - t_0}\right)^{n-1},$$
$$\alpha_u(t)M_1 = -\frac{2}{\gamma+1}\frac{r_0 n}{t_f - t_0}\left(\frac{t_f - t}{t_f - t_0}\right)^{n-1}. \qquad (12)$$

Requiring that $M_1 = \text{const}$ is independent of r_0, t_0, t_f, and n, we derive

$$M_1 = \frac{2}{\gamma+1}, \quad \alpha_u(t) = -\frac{r_0 n}{t_f - t_0}\left(\frac{t_f - t}{t_f - t_0}\right)^{n-1}. \qquad (13)$$

Let $t = t_0$ from (12) determine the n-dependence of t_f:

$$t_f = t_0 - \frac{2r_0 n}{(\gamma+1)u_{10}}.$$

The values of δ_1, Π_1, and dependences $\alpha_\rho(t)$ and $\alpha_p(t)$ are determined by analogy with M_1 and $\alpha_u(t)$:

$$\delta_1 = \frac{\gamma+1}{\gamma-1}, \quad \alpha_\rho = \rho_0,$$
$$\Pi_1 = \frac{2}{\gamma+1}, \quad \alpha_P = \rho_0 D_{10}^2\left(\frac{t_f - t}{t_f - t_0}\right)^{2(n-1)}. \qquad (14)$$

From (7), (13), and (14), we determine

$$\varphi_1 = 0, \quad \varphi_2 = \frac{n-1}{n}, \quad \varphi_3 = \frac{2(n-1)}{n}, \quad \omega = -1.$$

Let us substitute φ_1, φ_2, φ_3, and ω into Eqs. (5) and (6) and transform them to the form

$$M' = \frac{R_1 - 2\gamma n M\Pi}{R_2}, \quad \delta' = \frac{\delta(2M\delta n(M-\xi)^2 - R_1)}{R_2(M-\xi)}, \qquad (15)$$

$$\Pi' = \frac{\delta\Pi}{R_2}(2(n\gamma M + \xi(n-1))(M-\xi) - (n-1)\gamma\xi M), \qquad (16)$$

where

$$R_1 = (n-1)\xi((M-\xi)\delta M - 2\Pi),$$
$$R_2 = n\xi(\gamma\Pi - (M-\xi)^2\delta).$$

Functions $\delta(\xi)$, $\Pi(\xi)$, and $M(\xi)$ are found by integration of Eqs. (15) and (16) in region $1 \le \xi < \infty$. Value $\xi = \infty$ is attained at $t = t_f$ and $r > 0$. Quantity n can be found during the integration of Eqs. (15), (16) from the condition of simultaneous zeroing of the nominators in Eqs. (15), (16) and denominator R_2. Here we present the values of $n(\gamma)$ thus found:

γ	1.1	1.2	4/3	1.4	5/3
n	0.795973	0.757142	0.729259	0.717175	0.688377
δ_∞	184.465	59.5525	26.5447	20.0714	9.549680

The gas density in the focusing instant is independent of the radius and is determined by the relationship $\rho = \rho_0\delta_\infty$. Here, we also present the values of δ_∞ for various γ.

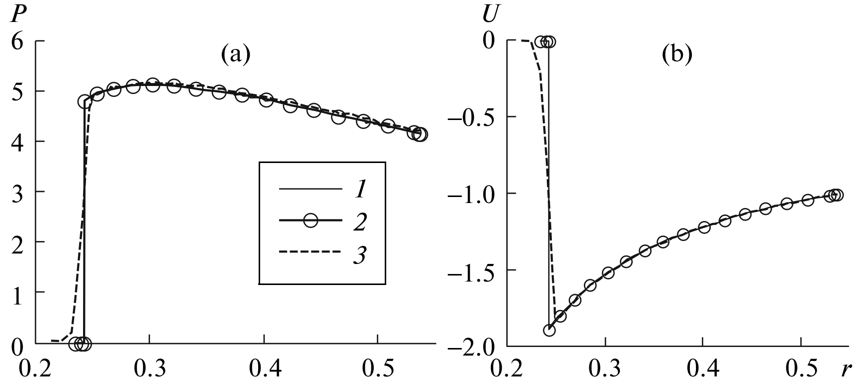

Fig. 1.

The described solution was applied to evaluate the accuracy of several computational methods of shock waves. A cold gas sphere with size $r_0 = 1$ had parameters $P_0 = 0$, $\rho_0 = 1$, $U_0 = 0$, $U_{01} = -1$, and $\gamma = 5/3$. The boundary condition was determined as follows. The sphere mass is independent of time. At instant t_*, it equals the sum m_1 and the gas mass between the SW and boundary

$$\frac{4}{3}\pi\rho_0 r_1^3 + \int_{r_1}^{r_*} 4\pi r^2 \rho dr = \frac{4}{3}\pi\rho_0 r_0^3. \tag{17}$$

Let us pass to integration over ξ. Dependence

$$r = r_0\xi\left(\frac{t_f - t_*}{t_f - t_0}\right)^n \tag{18}$$

follows from (10) at $t = t_*$. Substituting (18) into (17), we derive the equation for determining the coordinate of boundary ξ_*

$$1 + 3\int_1^{\xi_*} \delta\xi^2 d\xi - \left(\frac{t_f - t_0}{t_f - t_*}\right)^{3n} = 0.$$

$M(\xi)$ and $\Pi(\xi)$ are determined using ξ_* from dependences M_* and Π_*, and U and P at the sphere boundary are determined from (4). We find the boundary radius from (18) by ξ_*. The time dependence of the boundary velocity for $\gamma = 5/3$ is presented in the table.

Dependences (a) $P(r)$ and (b) $U(r)$ for instant $t_* = 0.45$ are presented in Figs. 1a and 1b; (*1*) is the analytical solution of this study and (*2*, *3*) are calculations by method [6] for the grid uniform by r with the number of points $N = 100$ ((*2*) with the isolation of the SW front and (*3*) without the isolation of the SW front).

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