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Collapse of Spherical Cavities and Energy Cumulation in an Ideal Compressible Liquid

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Abstract: Analytical solutions of the problem of collapsing of a spherical shell or cavity in an ideal compressible liquid having a constant density during its motion are constructed. The influence of the gas located in the cavity on the motion of the cavity boundary is studied. A quantitative characteristic of energy cumulation is proposed. An expression for energy cumulation in the case of shell or cavity collapsing is derived. The energy cumulation obtained in this study is compared with Zababakhin's results.

Keywords: cumulation, incompressibility, pressure, energy, velocity, energy release.

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INTRODUCTION

The problem of collapsing of bubbles in a liquid was first posed because of corrosion of marine propellers. The first solution of the problem of collapsing of a spherical bubble in an ideal incompressible liquid was obtained by Rayleigh in 1917. The problem revival was due to the development of nuclear weapons. Zababakhin constructed an analytical solution for focusing of a spherical shell made of an incompressible material under the action of an initial pulse. It was possible to publish this solution only in 1965 [1]. Collapsing of an empty cavity in an ideal compressible liquid was considered by Hunter [2]. The solution constructed by Hunter is physically meaningful only in a limited region far from focusing. A detailed analysis of self-similar solutions for implosion of a cavity in a gas can be found in the review of Kazhdan and Brushlinskii [3]. Later on, Zababakhin considered a collapse of a spherical bubble in a viscous liquid [4]. A detailed review of publications dealing with collapsing of spherical cavities can be found in [5].

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CONSERVATION LAWS AND SIMPLIFYING ASSUMPTIONS

The laws of conservation of mass, momentum, and energy in an ideal compressible liquid in the Lagrangian coordinates for a spherically symmetric case have the form

$$\frac{\partial V}{\partial t} - 4\pi\frac{\partial r^2 u}{\partial M} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + 4\pi r^2\frac{\partial p}{\partial M} = 0, \tag{2}$$

$$\frac{\partial}{\partial t}\left(E + \frac{1}{2}u^2\right) + 4\pi\frac{\partial r^2 pu}{\partial M} = 0. \tag{3}$$

Here V is the specific volume, p is the pressure, u is the velocity, r and M are the Eulerian and Lagrangian coordinates, t is the time, and E is the specific internal energy. Equations (1)–(3) yield the equation for the specific internal energy

$$\frac{\partial E}{\partial t} + p\frac{\partial V}{\partial t} = 0. \tag{4}$$

Equations (1)–(4) are written in the Lagrangian coordinates. The partial derivatives with respect to time in these equations are substantial derivatives because they are taken at a constant coordinate M, i.e., along the particle trajectory.

It is known from thermodynamics that the compressibility β_s and the velocity of sound C are determined by the equations

$$\beta_s = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_s, \quad C^2 = -V^2\left(\frac{\partial p}{\partial V}\right)_s.$$

Thus, the squared velocity of sound is inversely proportional to the compressibility:

$$C^2 = V/\beta_s.$$

There are no incompressible substances in nature. However, mechanics deals with a vast class of flows where the density remains constant in time. The constancy of the flow density is often recognized as incompressibility, i.e., $\beta_s = 0$ and $C^2 = \infty$. This is misguiding thinking. The property of the flow is not the property of the substance.

In a compressible liquid with a finite specific volume, we always have $C^2 > 0$ at $\beta_s > 0$, and system (1)–(3) has three characteristics. The assumption of incompressibility ($\beta_s = 0$) means that $C = \infty$ and system (1)–(3) is not hyperbolic. Both results contradict physical facts. The energy conservation law (3) and the equation of state are usually not considered in solving Eq. (2) in the model of an "incompressible" liquid [4]. For this reason, the model has internal contradictions. It follows from Eq. (4) that $E = \text{const}$ at $V = \text{const}$ in an ideal "incompressible" liquid. In this case, however, it follows from the equation of state $p = p(V, E)$ that $p = \text{const}$ as well. Nevertheless, system (1), (2) at $V = \text{const}$ has several solutions with changing p. Thus, the model has the following contradiction: on the one hand, the pressure changes, on the other hand, the pressure is constant. This contradiction cannot be eliminated within the framework of the model of an adiabatic "incompressible" medium, which included only Eqs. (1) and (2). However, this contradiction can be eliminated by assuming that the medium is not adiabatic, i.e., it contains sources of energy. In this case, Eq. (4) should be written in the following form:

$$\frac{\partial E}{\partial t} = -p\frac{\partial V}{\partial t} + \frac{\partial q}{\partial t}. \tag{5}$$

Let us present p and E as a sum of cold and thermal components:

$$p = p_{\text{c}}(V) + p_{\text{th}}, \quad E = E_{\text{c}}(V) + E_{\text{th}},$$
$$p_{\text{c}} = -\frac{dE_{\text{c}}}{dV}.$$

Substituting p and E into Eq. (5), we obtain the equation for the thermal pressure and thermal internal energy. In the general case, the change in E_{th} is determined by the dissipative function q and the work of the thermal pressure p_{th} with a change in the specific volume. At $V = \text{const}$, however, the work of the thermal pressure is equal to zero, and the change in E_{th} is determined only by the dissipative function q:

$$\frac{\partial E_{\text{th}}}{\partial t} = \frac{\partial q}{\partial t}.$$

It follows from the theory of equations of state [6] that the parameters p_{th} and E_{th} in a liquid are related by the equation $p_{\text{th}}V = \Gamma(V)E_{\text{th}}$. At $V = \text{const}$, this equation is used below in the form

$$pV_0 = \Gamma E, \tag{6}$$

where $\Gamma = \text{const}$, $p = p_{\text{th}}$, and $E = E_{\text{th}}$. As $p(t, M)$ is the solution of Eqs. (1), (2), then the dependence $E(t, M)$, which follows from Eq. (6), is fairly correct in each flow. It is determined by the necessity of satisfying the condition $V = \text{const}$.

One of the known dissipative functions is the viscosity. However, as each substance has a fairly particular viscosity coefficient, the viscosity is responsible only for some part of energy dissipation necessary for maintaining a constant density.

In this work, we confine ourselves to flows where the specific volume depends neither on M, nor on t, i.e., $V = \text{const}$. Simultaneously, the liquid is compressible: its compressibility β_s is not equal to zero.

GENERAL SOLUTION

At $V = V_0$, Eq. (1) reduces to the equation $\frac{\partial r^2 u}{\partial M} = 0$, which has the solution

$$r^2 u = f(t). \tag{7}$$

As f is independent of M, then Eq. (7) is valid for all values of M. On the bubble boundary, where $M = 0$, Eq. (7) takes the form

$$r_{\text{int}}^2 u_{\text{int}} = f(t), \tag{8}$$

where r_{int} is the coordinate of the internal surface and u_{int} is the velocity of the internal boundary of the bubble. Equations (7) and (8) yield the dependence of the velocity on the coordinate

$$u = u_{\text{int}} r_{\text{int}}^2 r^{-2}.$$

Let us express u_{int} from Eq. (8) and substitute it into the equation of the boundary trajectory $\frac{dr_{\text{int}}}{dt} = u_{\text{int}}$. Integrating this equation, we obtain

$$r_{\text{int}} = \left(r_{\text{int},0}^3 + \int_{t_0}^{t} 3f(t)dt\right)^{1/3}. \tag{9}$$

It is seen from Eqs. (8) and (9) that the cavity surface trajectory is completely determined by the function $f(t)$. If $f(t) < 0$, then $u_{\text{int}} < 0$; thus, the cavity collapses. The time of focusing of the cavity surface t_f is determined from Eq. (9) at $r_{int} = 0$:

$$r_{int,0}^3 + \int_{t_0}^{t_f} 3f(t)dt = 0. \quad (10)$$

Let us now consider Eq. (2). Using Eqs. (7) and (8), we transform Eq. (2) to

$$\frac{\partial p}{\partial M} = -\frac{1}{4\pi r^2}\left(\frac{f}{r^2 r_{int}^2}\frac{df}{dr_{int}} - \frac{2f^2}{r^5}\right). \quad (11)$$

Let us multiply Eq. (11) by $dM = 4\pi r^2 V_0^{-1} dr$ and integrate the resultant equation from 0 to M (from r_{int} to r):

$$p = p_{int}(t) + \frac{1}{V_0}\left[\frac{f}{r_{int}^2}\frac{df}{dr_{int}}\left(\frac{1}{r} - \frac{1}{r_{int}}\right) - \frac{f^2}{2}\left(\frac{1}{r^4} - \frac{1}{r_{int}^4}\right)\right]. \quad (12)$$

In this equation, r depends on t and M; $p_{int}(t)$ is the pressure on the cavity surface (at $M = 0$). The dependence $p_{int}(t)$ is determined either by the gas located inside the cavity or by the surface tension. If both these factors are absent, then $p_{int} = 0$. The limiting pressure (at $r = \infty$) is denoted by p_∞,

$$p_\infty(t) = p_{int}(t) - \frac{1}{V_0}\left(\frac{df}{dr_{int}}\frac{f}{r_{int}^3} - \frac{f^2}{2}\frac{1}{r_{int}^4}\right). \quad (13)$$

Among the four functions of time, $p_\infty(t)$, $p_{int}(t)$, $r_{int}(t)$, and $f(t)$, two arbitrary functions are independent. In other words, flows with two-functional arbitrariness are considered.

Using Eq. (13), we rewrite Eq. (12) as

$$p = p_\infty(t) + \frac{1}{V_0}\left(\frac{df}{dr_{int}}\frac{f}{r_{int}^2 r} - \frac{f^2}{2r^4}\right). \quad (14)$$

Passing from p to E with the use of Eq. (6) and differentiating Eq. (14) with respect to t, we obtain the equation for the energy dissipation rate

$$\frac{\partial q}{\partial t} = \frac{1}{\Gamma}\left(V_0\frac{dp_\infty(t)}{dt} + \frac{1}{r}\frac{d^2 f}{dt^2} - \frac{2f}{r^4}\frac{df}{dt} + \frac{2f^3}{r^7}\right). \quad (15)$$

Equations (7)–(15) have a general character. The variables r, u, p, E, $\frac{\partial p}{\partial M}$, and $\frac{\partial q}{\partial t}$ depend on t and M. All these dependences acquire a particular character after two of the four functions of time are defined: $p_\infty(t)$, $p_{int}(t)$, $r_{int}(t)$, and $f(t)$.

ENERGY CUMULATION

As there are several types of energy in mechanics of continuous media (internal, kinetic, cold, thermal, free energy, etc.), the notion of energy cumulation has to be specified. For focusing spherical cavities in an incompressible liquid, Zababakhin proposed the following definition of energy cumulation [1, 5, 7, 8]. At the time t_0, the maximum pressure is $\max p_0$; at the time $t > t_0$, its value is $\max p$. The ratio of these pressures was called energy cumulation in [1, 5, 7, 8]:

$$K = \max p / \max p_0. \quad (16)$$

The pressure reaches the maximum value on the line described by Eq. (11) at $\frac{\partial p}{\partial M} = 0$:

$$r_{max} = r_{int}\left(2f \Big/ r_{int}\frac{df}{dr_{int}}\right)^{1/3}. \quad (17)$$

Substituting r_{max} into Eq. (14), we obtain

$$\max p = p_\infty + \frac{3}{V_0}\left(\frac{df}{dr_{int}}\right)^{4/3} f^{2/3} r_{int}^{-8/3} 2^{-7/3}. \quad (18)$$

In the Rayleigh–Zababakhin solution, the level of energy cumulation unlimitedly increases as $r_{int} \to 0$ with a power-law exponent $n = 3$ as

$$K_p = G_1\left(\frac{r_{int,0}}{r_{int}}\right)^3, \quad (19)$$

where $G_1 = \text{const}$ at $r_{int} = 0$. This interesting theoretical result was not validated by practice: it was impossible to reach unlimited energy cumulation in experiments. Zababakhin's studies [5] showed that theoretical unlimited energy accumulation is not constrained if the real conditions (including energy dissipation due to viscosity and thermal conductivity) are taken into account to the maximum possible extent, and the question about the factor limiting energy cumulation remains open. Probably, energy cumulation is limited by instability [5]. Certainly, instability is a limiting factor for energy cumulation. Nevertheless, there are also other reasons.

All dependences $K_p(r_{int})$ in flows with $V = \text{const}$ were obtained with the energy equation and the equation of state being ignored. In other words, the energy that should be additionally spent for satisfying the condition $V = \text{const}$ in the flow was ignored.

Let us change the definition of energy cumulation by introducing a coefficient K_E, which is the ratio of the maximum specific internal energy E to the mean energy E_m averaged over the domain M_0:

$$K_E = \max E / E_m, \quad (20)$$

where

$$E_m = \frac{Q}{M_0}, \quad Q = \int_0^{M_0} E\,dM. \quad (21)$$

The mass M_0 is determined at the time $t = t_0$ as the mass between the cavity boundary and the point $r_{\max 0}$ determined by Eq. (17), where the energy $E(r)$ reaches the maximum value.

Let us assume that energy cumulation is unlimited if $K \to \infty$ as $r_{\text{int}} \to 0$. If $K \to K_1$, where $1 \ll K_1 < \infty$, then energy cumulation is limited. In all other cases, there is no cumulation.

The physical essence of this definition of energy cumulation means that the character of the solution ensures a significantly greater maximum value than the mean value at least at one point regardless of the degree of increasing of the mean specific internal energy with time.

Substituting E from Eqs. (6) and (14) into Eq. (21), integrating the resultant equation and dividing it by M_0, we determine the specific internal energy averaged over the mass M_0:

$$E_m = \frac{p_\infty V_0}{\Gamma} + \frac{4\pi}{2\Gamma M_0 V_0} \times \left[\frac{df}{dr_{\text{int}}}\frac{f}{r_{\text{int}}^2}(r_a^2 - r_{\text{int}}^2) + f^2\left(\frac{1}{r_a} - \frac{1}{r_{\text{int}}}\right)\right]. \quad (22)$$

Here r_a is the external coordinate of a spherical layer of mass M_0. The dependence $E(r)$ is nonmonotonic, and $\max E = V_0 \max p \cdot \Gamma^{-1}$ is reached on the line $r_{\max}(t)$ (17).

To find a particular expression for K, we have to eliminate arbitrariness in choosing the functions $f(t)$, $p_{\text{int}}(t)$, $r_{\text{int}}(t)$, and $p_\infty(t)$. For example, in the Rayleigh–Zababakhin flow with $p_{\text{int}}(t) = 0$ and $p_\infty(t) = 0$, the function $f(t)$ has the form $f = u_{\text{int},0} r_{\text{int},0}^{3/2} r_{\text{int}}^{1/2}$. With the energy-based approach to the energy cumulation definition in this solution, K_E is determined from Eqs. (20), (22), and (18):

$$K_E \approx G_2\left(\frac{r_a}{r_{\text{int}}}\right), \quad (23)$$

where $G_2 = \text{const}$ at $r_{\text{int}} = 0$. A comparison of Eqs. (23) and (19) shows that the power-law exponent of the energy cumulation coefficient decreases by a factor of 3 as $r_{\text{int}} \to \infty$.

CLASS OF THE SIMPLEST SOLUTIONS

We take the function $f(t)$ in the form of the power-law dependence $f = dr_{\text{int}}^\alpha$, where d and α are constants. As $d = \text{const}$, then it follows from Eq. (8) at $t = t_0$ that $d = u_{\text{int},0} r_{\text{int},0}^{2-\alpha}$. Thus, the class of solutions is partly determined by the function $f(t)$ of the form

$$f = u_{\text{int},0} r_{\text{int},0}^{2-\alpha} r_{\text{int}}^\alpha \quad (24)$$

with one constant value of α. The chosen form of f (24) corresponds to

$$\frac{df}{dt} = \alpha u_{\text{int},0}^2 r_{\text{int},0}^{2(2-\alpha)} r_{\text{int}}^{2\alpha-3}, \quad (25)$$

$$u_{\text{int}} = u_{\text{int},0} r_{\text{int},0}^{2-\alpha} r_{\text{int}}^{\alpha-2}, \quad u = u_{\text{int},0} r_{\text{int},0}^{2-\alpha} r_{\text{int}}^\alpha r^{-2}. \quad (26)$$

For complete definition of the class of solutions, we confine ourselves to the case where the pressure on the cavity boundary is equal to zero: $p_{\text{int}} = 0$. In this case, Eqs. (13) and (24) yield the equation

$$p_\infty = \frac{1}{V_0} u_{\text{int},0}^2 r_{\text{int},0}^{2(2-\alpha)} r_{\text{int}}^{2(\alpha-2)}\left(\frac{1}{2} - \alpha\right). \quad (27)$$

As the thermal pressure cannot be negative even at infinity, there is a constraint on the domain of possible values of α: $\alpha \leqslant 1/2$. For the class of solutions considered here, the dependence $p(r)$ has the form

$$p = \frac{u_{\text{int},0}^2}{V_0}\left(\frac{r_{\text{int},0}}{r_{\text{int}}}\right)^{2(2-\alpha)} \times \left[\frac{1}{2}\left(1 - \frac{r_{\text{int}}^4}{r^4}\right) - \alpha\left(1 - \frac{r_{\text{int}}}{r}\right)\right]. \quad (28)$$

Let us first find the energy cumulation coefficient K_p. For this purpose, we determine $\max p$. At the point $r_{\max} = r_{\text{int}}(2/\alpha)^{1/3}$, Eq. (28) yields

$$\max p = \frac{u_{\text{int},0}^2}{2V_0}\left(\frac{r_{\text{int},0}}{r_{\text{int}}}\right)^{2(2-\alpha)} \times \left[1 - 2\alpha - \left(\frac{\alpha}{2}\right)^{4/3} + 2\alpha\left(\frac{\alpha}{2}\right)^{1/3}\right]. \quad (29)$$

At $r_{\text{int}} = r_{\text{int},0}$, Eq. (29) yields $\max p_0$. Substituting $\max p$ and $\max p_0$ into Eq. (16), we obtain

$$K_p \approx G_3\left(\frac{r_{\text{int},0}}{r_{\text{int}}}\right)^{2(2-\alpha)}, \quad (30)$$

where $G_3 = \text{const}$ at $r_{\text{int}} = 0$. At $\alpha = 1/2$, the energy cumulation coefficient K_p (30) coincides with that obtained by Zababakhin; at $\alpha < 1/2$, it increases.

Let us now determine the cumulation coefficient by the energy-based method. We multiply p (28) by V_0/Γ, substitute the resultant expression for E into Eq. (21), and determine the total internal energy of the mass M_0. Integrating E with respect to the mass M_0, we find the internal energy of this mass

$$Q = \frac{4\pi u_{\text{int},0}^2}{3\Gamma V_0}\left(\frac{r_{\text{int},0}}{r_{\text{int}}}\right)^{2(2-\alpha)} F_1, \quad (31)$$

where

$$F_1 = \frac{1}{2}(1 - 2\alpha) r_a^3 + r_{\text{int}}\left[\frac{3}{2}\alpha r_a^2 + r_{\text{int}}^2\left(\frac{3}{2}\frac{r_{\text{int}}}{r_a} - \frac{1}{2}\alpha - 2\right)\right]. \quad (32)$$

At the initial time $t = t_0$, at $r_{\text{int}} = r_{\text{int},0}$, the total energy of the mass M_0 is Q_0, and $F_1 = F_{10}$. Dividing Q by Q_0, we obtain

$$\frac{Q}{Q_0} = \left(\frac{r_{\text{int},0}}{r_{\text{int}}}\right)^{2(2-\alpha)} \frac{F_1}{F_{10}}.$$

It is seen from Eq. (29) that $\max p \to \infty$ as $r_{\text{int}} \to 0$, regardless of r; therefore, $\max E \to \infty$, as the total energy Q with the power-law exponent $2(2-\alpha)$.

The specific internal energy averaged over the mass M_0 is obtained by dividing Q (31) by M_0; $\max E$ is reached on the line $r_{\max}(t)$ (17) and is expressed via $\max p$ as follows:

$$\max E = \frac{V_0}{\Gamma} \max p. \tag{33}$$

It follows from Eqs. (20), (29), (31), and (33) that

$$K_E = \frac{\Gamma V_0 M_0}{4\pi F_1}. \tag{34}$$

It is seen from Eqs. (32) and (34) that there is no energy cumulation at $\alpha < 1/2$. At $\alpha = 1/2$, the first term in F_1 vanishes. In this case, first-order cumulation exists as $r_{\text{int}} \to 0$. It should be noted that the solution of the considered class at $\alpha = 1/2$ coincides with the Rayleigh–Zababakhin solution:

$$p_{\text{int}} = 0, \quad p_\infty = 0, \quad u_{\text{int}} = u_{\text{int},0}\left(\frac{r_{\text{int},0}}{r_{\text{int}}}\right)^{3/2}.$$

As an example, let us determine the trajectory of the cavity boundary for the solution in which expressions (24) and (26) take the following form at $\alpha = 0$:

$$f = u_{\text{int},0} r_{\text{int},0}^2, \quad u_{\text{int}} = u_{\text{int},0} r_{\text{int},0}^2 r_{\text{int}}^{-2}. \tag{35}$$

Substituting f from Eq. (35) into Eq. (9) and integrating the resultant expression, we obtain

$$r_{\text{int}} = [r_{\text{int},0}^3 + 3u_{\text{int},0} r_{\text{int},0}^2 (t - t_0)]^{1/3}.$$

The instant of focusing is determined from this equation at $r_{\text{int}} = 0$:

$$t_{\text{f}} = t_0 - \frac{r_{\text{int},0}}{3u_{\text{int},0}}. \tag{36}$$

With the help of Eq. (36), the equation of the bubble boundary trajectory is transformed to

$$r_{\text{int}} = r_{\text{int},0}\left(\frac{t_{\text{f}} - t}{t_{\text{f}} - t_0}\right)^{1/3}.$$

COLLAPSE OF A GAS-CONTAINING CAVITY

The gas located in the bubble decelerates the motion of the bubble boundary. We assume that the density of the gas in the bubble depends only on its volume: $\rho = \rho_0 (r_{\text{int},0}/r_{\text{int}})^3$. We also assume that the gas is compressed isentropically. The isentrope equation for an ideal gas yields

$$p_{\text{int}} = p_{\text{int},0}(r_{\text{int},0}/r_{\text{int}})^{3\gamma}.$$

As the liquid already moves toward the center of symmetry at $t = t_0$, the value of $p_{\text{int},0}$ is not arbitrary. It should be correlated with the state of the liquid on the cavity surface. To simplify the procedure of constructing the analytical solution, we consider only the case with $\gamma = 4/3$. Let us also assume that $p_\infty = 0$. Thus, the function $f(t)$ becomes dependent on p_∞ and p_{int}. It is determined from Eq. (13) and has the form

$$f = u_{\text{int},0} r_{\text{int},0}^2 \sqrt{\frac{r_{\text{int}} - r_{\text{int,cav}}}{r_{\text{int},0} - r_{\text{int,cav}}}}, \tag{37}$$

where $r_{\text{int,cav}}$ is the minimum value of the cavity radius at which the velocity of its boundary vanishes. As $p_\infty = 0$, Eqs. (14) and (37) yield the following expression at the time t_0:

$$p_{\text{int},0} = \frac{u_{\text{int},0}^2 r_{\text{int,cav}}}{2V_0(r_{\text{int},0} - r_{\text{int,cav}})}.$$

From Eqs. (8) and (37), we obtain the dependence $u_{\text{int}}(t)$:

$$u_{\text{int}} = u_{\text{int},0}\left(\frac{r_{\text{int},0}}{r_{\text{int}}}\right)^2 \sqrt{\frac{r_{\text{int}} - r_{\text{int,cav}}}{r_{\text{int},0} - r_{\text{int,cav}}}}. \tag{38}$$

The cavity boundary trajectory is determined by integrating the equation $\dfrac{dr_{\text{int}}}{dt} = u_{\text{int}}$ together with Eq. (38):

$$A(r_{\text{int}}) - A(r_{\text{int},0}) = \frac{u_{\text{int},0} r_{\text{int},0}^2}{\sqrt{r_{\text{int},0} - r_{\text{int,cav}}}}(t - t_0),$$

where

$$A(r_{\text{int}}) = \frac{2}{15}(3r_{\text{int}}^2 + 4r_{\text{int}} r_{\text{int,cav}} + 8r_{\text{int,cav}}^2) \times \sqrt{r_{\text{int}} - r_{\text{int,cav}}}.$$

Under the condition $r_{\text{int}} = r_{\text{int,cav}}$, this equation determines the time instant of the maximum compression of the gas (time instant when the boundary ceases to move):

$$t_{\text{f}} = t_0 - \frac{2(r_{\text{int},0} - r_{\text{int,cav}})}{15u_{\text{int},0} r_{\text{int},0}^2} (3r_{\text{int},0}^2 + 4r_{\text{int},0} r_{\text{int,cav}} + 8r_{\text{int,cav}}^2).$$

At each fixed time instant, the dependence of the liquid velocity on time is determined by Eqs. (7) and (37). The dependence $p(r, t)$ is determined from Eqs. (14) and (37):

$$p(r,t) = \frac{u_{\rm int,0}^2 r_{\rm int,0}^4}{2V_0 r_{\rm int}^2 (r_{\rm int,0} - r_{\rm int,cav})} \times \left[\frac{1}{r} - \frac{(r_{\rm int} - r_{\rm int,cav}) r_{\rm int}^2}{r^4}\right]. \quad (39)$$

The maximum value of p is reached at the value of $r_{\max}$ satisfying the equation

$$\left(\frac{r_{\max}}{r_{\rm int}}\right)^3 = 4\left(1 - \frac{r_{\rm int,cav}}{r_{\rm int}}\right).$$

It follows from this equation that $r_{\max} < r_{\rm int}$ at $r_{\rm int,cav} \geqslant 0.75 r_{\rm int}$; thus, the pressure reaches the maximum value on the cavity boundary. If the classical criterion of energy cumulation is applied, it is necessary to determine the value of $\max p_0$, which is reached at $r_{\max 0} = r_{\rm int,0}[4(1 - r_{\rm int,cav}/r_{\rm int,0})]^{1/3}$. At this point, we have

$$\max p_0 = \frac{u_{\rm int,0}^2 r_{\rm int,0}^{4/3}}{V_0 (r_{\rm int,0} - r_{\rm int,cav})^{4/3} \cdot 2^{11/3}}.$$

The maximum pressure reached at the boundary at the time t is

$$p_{\rm int} = \frac{u_{\rm int,0}^2 r_{\rm int,0}^4 r_{\rm int,cav}}{2V_0 (r_{\rm int,0} - r_{\rm int,cav}) r_{\rm int}^4}. \quad (40)$$

The ratio $p_{\rm int}/\max p_0$ determines the value of K_p. At the instant of the maximum compression of the gas bubble, we have

$$K_p = 2^{8/3} (r_{\rm int,0} - r_{\rm int,cav})^{1/3} r_{\rm int,0}^{8/3} r_{\rm int,cav}^{-3}.$$

Thus, limited energy cumulation is obtained.

Let us now apply the energy-based criterion for estimating energy cumulation. In accordance with Eq. (6), E is proportional to p (39); therefore, the maximum value of E at $r_{\rm int,cav} > 0.75 r_{\rm int}$ is reached on the cavity boundary. The value of the mean energy E_m averaged over the mass M_0 is obtained from Eqs. (21) and (39) after multiplication of p by V_0/Γ:

$$E_m = \frac{4\pi u_{\rm int,0}^2 r_{\rm int,0}^4}{2\Gamma V_0 M_0 (r_{\rm int,0} - r_{\rm int,cav}) r_{\rm int}^2} \times \left[\frac{1}{2}(r_a^2 - r_{\rm int}^2) + (r_{\rm int} - r_{\rm int,cav}) r_{\rm int}^2 \left(\frac{1}{r_a} - \frac{1}{r_{\rm int}}\right)\right]. \quad (41)$$

Substituting Eqs. (40) and (41) into Eq. (20), we obtain the expression for K_E as $r_{\rm int} \to r_{\rm int,cav}$:

$$K_E = \frac{V_0 M_0}{2\pi r_{\rm int,cav}} \times \left[\left(r_{\rm int,cav}^3 + \frac{4V_0}{3\pi} M_0\right)^{2/3} - r_{\rm int,cav}^2\right]^{-1}. \quad (42)$$

It is seen from Eq. (42) that energy cumulation is limited in accordance with the energy-based criterion. The difference in the results obtained by applying two different criteria can be summarized as follows: as $r_{\rm int,cav}$ decreases, the energy cumulation coefficient K_p increases as $r_{\rm int,cav}^{-3}$, whereas the coefficient K_E increases as $r_{\rm int,cav}^{-1}$.

FOCUSING OF THE SHELL

Following [1, 5], we consider focusing of an unloaded shell with zero values of pressure on the internal and external boundaries. This means that the ambient medium does not perform any work on the shell, and all internal processes in the shell are determined only by the initial energy and energy release. At the initial time t_0, the radius $r_{\rm int,0}$ and velocity $u_{\rm int,0} < 0$ of the internal boundary are defined. In accordance with Eqs. (7) and (8), the velocity in the shell at the time instant t_0 depends on the radius:

$$u = u_{\rm int,0} (r_{\rm int,0}/r)^2. \quad (43)$$

All parameters on the external boundary of the shell will be indicated by the subscript a. Let the shell mass be given and indicated by M_a. The coordinate of the external surface r_a is related to the coordinate of the internal surface $r_{\rm int}$ by the equation

$$r_a = (r_{\rm int}^3 + b)^{1/3}, \quad (44)$$

where $b = [3V_0/(4\pi)] M_a$. The initial velocity of the external boundary is found from Eqs. (43) and (44):

$$u_{a0} = u_{\rm int,0} (r_{\rm int,0}/r_{a0})^2.$$

At $p_a = 0$ and $p_{\rm int} = 0$, Eq. (12) takes the form

$$\frac{1}{r_{\rm int}^2} \frac{df}{dr_{\rm int}} \left(\frac{1}{r_a} - \frac{1}{r_{\rm int}}\right) - \frac{f}{2}\left(\frac{1}{r_a^4} - \frac{1}{r_{\rm int}^4}\right) = 0. \quad (45)$$

Let us now decompose the difference of the fourth-power terms into the difference of the first-power terms and an incomplete cube of the sum. As a result, Eq. (45) transforms to

$$\frac{df}{dr_{\rm int}} = \frac{f}{2}\left(\frac{r_{\rm int}^2}{r_a^3} + \frac{r_{\rm int}}{r_a^2} + \frac{1}{r_a} + \frac{1}{r_{\rm int}}\right).$$

Let us consider the solution of this equation:

$$\ln f = \ln \Omega + \frac{1}{2}(J_1 + J_2 + J_3 + J_4),$$

where

$$J_1 = \int \frac{r_{\rm int}^2 dr_{\rm int}}{r_{\rm int}^3 + b}, \quad J_2 = \int \frac{r_{\rm int} dr_{\rm int}}{(r_{\rm int}^3 + b)^{2/3}},$$

$$J_3 = \int \frac{dr_{\rm int}}{(r_{\rm int}^3 + b)^{1/3}}, \quad J_4 = \int \frac{dr_{\rm int}}{r_{\rm int}}.$$

The integrals J_1 and J_4 are tabular integrals, whereas J_2 and J_3 are binomial integrals. After integration, we obtain the dependence

$$f = \Omega (r_{\rm int}^3 + b)^{1/6} r_{\rm int}^{1/2} [(r_{\rm int}^3 + b)^{1/3} - r_{\rm int}]^{-1/2}. \quad (46)$$

The constant of integration Ω is found at $t = t_0$, where $f_0 = u_{\rm int,0} r_{\rm int,0}^2$ and $r_{\rm int} = r_{\rm int,0}$:

$$\Omega = u_{\rm int,0} r_{\rm int,0}^{3/2} [(r_{\rm int,0}^3 + b)^{1/3} - r_{\rm int,0}]^{1/2} (r_{\rm int,0}^3 + b)^{-1/6}.$$

For the function $f(r_{\rm int}(t))$, we determine the derivative

$$\frac{df}{dr_{\rm int}} = \frac{\Omega[(r_{\rm int}^3+b)^{4/3} - r_{\rm int}^4]}{2r_{\rm int}^{1/2}[(r_{\rm int}^3+b)^{1/3} - r_{\rm int}]^{3/2}(r_{\rm int}^3+b)^{5/6}}. \quad (47)$$

We write Eqs. (46) and (47) in the form

$$f = r_{\rm int}^{1/2} F_3(r_{\rm int}, r_a), \quad \frac{df}{dr_{\rm int}} = r_{\rm int}^{-1/2} F_4(r_{\rm int}, r_a), \quad (48)$$

where

$$\begin{aligned} F_3 &= \Omega r_a^{1/2}(r_a - r_{\rm int})^{-1/2}, \\ F_4 &= \frac{1}{2}\Omega r_a^{1/2} D (r_a - r_{\rm int})^{-1/2}, \end{aligned} \quad (49)$$

$$D = 1 + \frac{r_{\rm int}}{r_a} + \left(\frac{r_{\rm int}}{r_a}\right)^2 + \left(\frac{r_{\rm int}}{r_a}\right)^3. \quad (50)$$

The dependence $p(r)$ in the shell determined by Eq. (12) at $p_{\rm int} = 0$, f (46), and $\frac{df}{dr_{\rm int}}$ (47) has a maximum on the line

$$r_{\max} = 2^{2/3} r_{\rm int} D^{-1/3}. \quad (51)$$

Substituting $r_{\max}$ together with D, F_3, F_4, f, b, and $\frac{df}{dr_{\rm int}}$ from Eqs. (48)–(50) into Eq. (12), we obtain

$$\max p = r_{\rm int}^{-3}\frac{F_3}{V_0}\left[F_4(2^{-2/3}D^{1/3} - 1) - \frac{1}{2}F_3(2^{8/3}D^{4/3} - 1)\right]. \quad (52)$$

At the initial time instant $t = t_0$, the maximum value of p_0 is determined as

$$\max p_0 = r_{\rm int,0}^{-3}\frac{F_{30}}{V_0}\left[F_{40}(2^{-2/3}D_0^{1/3} - 1) - \frac{1}{2}F_{30}(2^{8/3}D_0^{4/3} - 1)\right].$$

As $r_{\rm in} \to 0$, the variables F_3, F_4, and D are limited; thus, we have

$$K_p \approx G_4\left(\frac{r_{\rm int,0}}{r_{\rm int}}\right)^3, \quad (53)$$

where $G_4 = \text{const}$ at $r_{\rm int} = 0$. Energy cumulation occurs because the cumulation coefficient K_p tends to infinity with a power-law exponent equal to 3. This result coincides with that obtained in [1, 5, 7, 8], though it was obtained by a different method.

During the shell motion period from t_0 to $t_{\rm f}$, its thermal energy increases. Integrating E with respect to M from $M = 0$ to $M_0 = M_a$, we obtain the expression for the internal energy of the shell

$$Q = F_5 r_{\rm int}^{-2}, \quad (54)$$

where

$$F_5 = \frac{\pi\Omega^2 r_a}{3V_0\Gamma(r_a - r_{\rm int})}\left[r_a^2\left(D + 8\frac{r_{\rm int}^3}{r_a^3}\right) - r_{\rm int}^2(D+8)\right].$$

As $r_{\rm int} \to 0$, the value of F_5 tends to a constant value:

$$\lim_{r_{\rm int}\to 0} F_5 = \frac{\pi\Omega^2 r_a^2}{3V_0\Gamma}.$$

Therefore, $Q \to \infty$ as $r_{\rm int} \to 0$ and

$$\frac{Q}{Q_0} \approx F_6\left(\frac{r_{\rm int,0}}{r_{\rm int}}\right)^2, \quad (55)$$

where $F_6 = \text{const}$. In other words, to satisfy the condition $V = \text{const}$ in the flow, it is necessary to spend a lot of energy. As the shell is a mechanically insulated system, there arises the following question: How can energy with a prescribed profile be imparted to the shell? Methods of energy input to the system are outside the scope of this work. It is important that the amount of energy imparted to the shell increases with time and tends to infinity in accordance with Eq. (55). As the internal energy is thermal energy, the temperature increases as this energy increases, and emission becomes essential after a certain time. All attempts to further increase the shell energy will be accompanied by its reduction owing to outward emission of radiation from the shell.

Formally, applying the energy-based criterion, we can conclude that theoretically energy cumulation exists, though its order differs from than predicted in [1, 5, 7, 8].

The mean value E_m is determined by dividing Q (54) by the shell mass M_a. Substituting $\max p$ from Eq. (52) into Eq. (33) and then $\max E$ and E_m into Eq. (20), we obtain

$$K_E \approx G_5\left(\frac{r_{\rm int,0}}{r_{\rm int}}\right),$$

where $G_5 = \text{const}$. This energy cumulation coefficient differs by two orders of magnitude from K_p (53) obtained by the classical method.

CONCLUSIONS

1. The model of an incompressible viscous liquid was developed by Navier and Stokes in 1820–1850. For spherically symmetric motion without viscosity, it reduces to Eqs. (2) and (9). The model contained neither the energy equation, nor the equation of state, which were added many years later. Owing to its ability to construct analytical solutions, the model became very popular among mechanics and mathematicians and still is.

2. The complete model of mechanics of continuous media (Euler–Helmholtz equations (1)–(3)) allows one to construct solutions with a constant density. Energy input is required to maintain a constant density in the liquid. This fact has been ignored until recently.

3. The new definition of energy cumulation, which is given in this paper and takes into account additional energy expenses, shows that cumulation drastically decreases in a number of flows and even vanishes in some of them.

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