ON A DIFFERENCE METHOD FOR THE CALCULATION OF SHOCK WAVES*

V. F. KUROPATENKO

(Chelyabinsk)

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The numerical integration of the equations of gas dynamics:

$$\left.\begin{aligned} &u_t = -P_x, \\ &V_t = u_x, \\ &\mathscr{E}_t = -(Pu)_x, \\ &\mathscr{E} = E + \tfrac{1}{2}u^2, \\ &P = f(V,E) \end{aligned}\right\} \tag{1}$$

is associated with a great many difficulties caused by the presence of lines in the x, t plane along which either the unknown functions or their derivatives suffer a discontinuity. The values of the functions on both sides of a strong discontinuity which is the front of a shock wave and is propagated with velocity W are connected by certain Rankine-Hugoniot relations:

$$\left.\begin{aligned} &\bar{V} - V_0 = -\frac{1}{W}(\bar{u} - u_0), \\ &\bar{u} - u_0 = \frac{1}{W}(\bar{P} - P_0), \\ &\bar{E} - E_0 = \tfrac{1}{2}(\bar{P} + P_0)(V_0 - \bar{V}). \end{aligned}\right\} \tag{2}$$

If we take into account the fact that the entropy of a particle increases only in the case when its trajectory intersects the line of strong discontinuity, and in the other cases remains unchanged along its path, we can divide all the solutions of system (1) into two types:

a) solutions with a dissipation of energy (strong discontinuities),

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b) solutions without dissipation of energy (weak discontinuities, contact discontinuities, continuous solutions).

In order to overcome the difficulties associated with the existence of strong discontinuities, various methods have been suggested which reduce to the methods of "through" computation [1-4]. The common feature of these methods is the elimination of the strong discontinuities, thus basically simplifying the calculations.

To calculate such blurred shock waves it is suggested in [3] that we use relations (2) in which all the quantities with a zero suffix and one of the quantities with a bar are assumed to be known. This enables us to determine all the other unknowns with a bar, and these play an auxiliary role and help us to find the approximate solution of system (1).

In this note we propose to use relations (2) to calculate blurred shock waves, as in [3]. But by choosing a net which is different from the net in [3] and selecting the known quantities in (2) in a different way we arrive at a new, purely divergent difference scheme.

Let us divide the region of integration of system (1) by a net over the intervals $h = x_{i+1} - x_i$ ($i = 0, 1, 2, \ldots, N-1$). We shall find all the unknowns at the points $x_{i+1/2}$. For the numerical integration we use the following system of difference equations:

$$u^{n+1}_{i+1/2} - u^n_{i+1/2} = -\frac{\tau}{h}(P^*_{i+1} - P^*_i), \tag{3}$$

$$V^{n+1}_{i+1/2} - V^n_{i+1/2} = \frac{\tau}{h}(U^*_{i+1} - U^*_i), \tag{4}$$

$$\mathscr{E}^{n+1}_{i+1/2} - \mathscr{E}^n_{i+1/2} = -\frac{\tau}{h}(P^*_{i+1}U^*_{i+1} - P^*_i U^*_i), \tag{5}$$

$$\mathscr{E}^{n+1}_{i+1/2} = E^{n+1}_{i+1/2} + \frac{1}{2}(u^{n+1}_{i+1/2})^2, \tag{6}$$

$$P^{n+1}_{i+1/2} = f(V^{n+1}_{i+1/2}, E^{n+1}_{i+1/2}). \tag{7}$$

The quantities U^*_i and P^*_i on the right-hand sides of equations (3)-(5) are unknown, since the approximate solution of system (1) is determined at the points $x_{i+1/2}$. To find them we use the following algorithm.

We shall call the approximate solution in the interval $[x_{i+1/2}, x_{i-1/2}]$ an *R*-wave if the condition

$$\frac{u_{i+1/2} - u_{i-1/2}}{x_{i+1/2} - x_{i-1/2}} \geqslant 0, \tag{8}$$

is fulfilled, and we call it an *S*-wave if

$$\frac{u_{i+1/2} - u_{i-1/2}}{x_{i+1/2} - x_{i-1/2}} < 0. \tag{9}$$

It is easy to show, using Tsemlen's theorem [5] and Taylor's formula, that for sufficiently small $h = x_{i+1/2} - x_{i-1/2}$ all the strong discontinuities are contained in the class of S-waves.

In the case of an R-wave U_i^* and P_i^* can be found by interpolation using the formulae

$$U_i^* = \frac{1}{2}(u_{i+1/2}^{n+1} + u_{i-1/2}^{n+1}),$$
$$P_i^* = \frac{1}{2}(P_{i+1/2}^{n} + P_{i-1/2}^{n}). \tag{10}$$

In the case of an S-wave U_i^* and P_i^* are set equal to $\bar{u}$ and $\bar{P}$. The value of $\bar{u}$ is found after solving system (2) together with the equation of state $P = f(V, E)$. We choose the known values in (2) and $\bar{P}$ in the following way:

a) $P_0 = P_{i-1/2}^n$, $u_0 = u_{i-1/2}^n$, $V_0 = V_{i-1/2}^n$, $E_0 = E_{i-1/2}^n$, $\bar{P} = P_{i+1/2}^n$ for $P_{i+1/2}^n > P_{i-1/2}^n$ and

b) $P_0 = P_{i+1/2}^n$, $u_0 = u_{i+1/2}^n$, $V_0 = V_{i+1/2}^n$, $E_0 = E_{i+1/2}^n$, $\bar{P} = P_{i-1/2}^n$ for $P_{i+1/2}^n < P_{i-1/2}^n$. If the equation of state has the form

$$P = (\gamma - 1) E \frac{1}{V}, \tag{11}$$

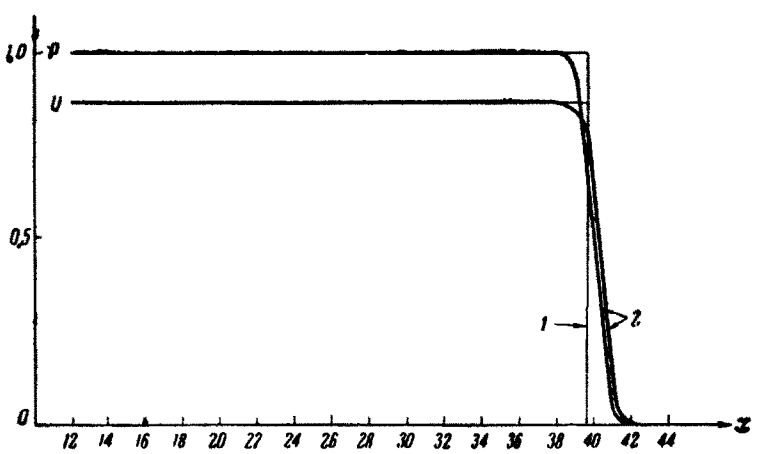

FIG. 1. 1 – exact solution, 2 – approximate solution.

$W > 0$ and the direction of increasing i coincides with the positive direction of the Ox axis, then to determine U_i^* and P_i^* we obtain the expressions

$$P_i^* = P^n_{i-1/2},$$
$$U_i^* = u^n_{i+1/2} + \frac{P^n_{i-1/2} - P^n_{i+1/2}}{\sqrt{\frac{1}{2V^n_{i+1/2}} \times [(\gamma+1) P^n_{i-1/2} - (\gamma-1) P^n_{i+1/2}]}}. \tag{12}$$

The choice of the direction of increasing i and the sign of W is not a restriction. For a different direction of increasing i or for $W < 0$ we will obtain expressions of the same type as (12) for U_i^* and P_i^* but with different suffixes.

In order to compare this method with previous ones, let us change from the finite-difference representation of the auxiliary functions U_i^*, P_i^* to their differential form. Assuming the existence of the first derivatives of the unknown functions and ignoring terms of order h^2 in the expansions, we obtain from (3)-(5) and (12)

$$\begin{aligned} u_t &= -P_x, \\ V_t &= U^*_x, \\ \mathscr{E}_t &= -(PU^*)_x, \end{aligned} \tag{13}$$

where $U^* = u + w$ and w is a small quantity of order h defined by the expression

$$w = -\frac{\frac{\partial P}{\partial x} \times h}{\sqrt{\frac{\gamma P}{V} - \frac{\gamma+1}{4} \frac{\partial P}{\partial x} \times h}}. \tag{14}$$

In the two limiting cases formula (14) takes the form

$$w = \begin{cases} -\dfrac{h}{c} \dfrac{\partial P}{\partial x} & \text{for } \dfrac{\gamma P}{V} \gg -\dfrac{\gamma+1}{4} \dfrac{\partial P}{\partial x} h, \\ \sqrt{\dfrac{4h}{\gamma+1} \dfrac{\partial P}{\partial x}} & \text{for } \dfrac{\gamma P}{V} \ll -\dfrac{\gamma+1}{4} \dfrac{\partial P}{\partial x} h, \end{cases} \tag{15}$$

where c is the mass sound velocity defined by the formula

$$c^2 = -\left(\frac{\partial P}{\partial V}\right)_s. \tag{16}$$

We can treat the quantity w as the "viscosity" introduced in the initial equations.

The use of formulae (12) to calculate U_i^* and P_i^* on an S-wave leads

to a satisfactory description of the dissipative process; the increment in the entropy of a particle as it passes across a blurred shock wave agrees well with the jump in entropy on the exact shock wave.

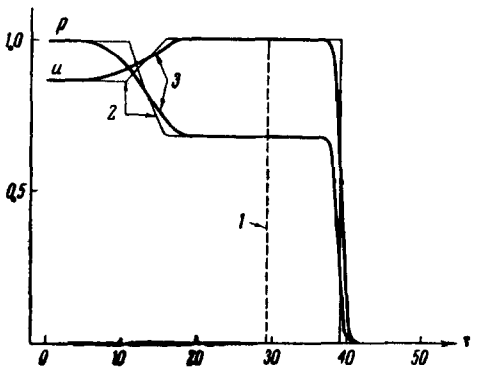

FIG. 2. *1* – contact discontinuity, *2* – exact solution, *3* – approximate solution.

Let us use our method for an arbitrary equation of state. In this case we can use a rapidly converging iteration process. The results can be generalised without difficulty to the case of a non-uniform net, and also to the case of flow with cylindrical or spherical symmetry.

The stability condition in this case has been established experimentally:

$$K = \frac{\tau C}{h} \leqslant 1.$$

Using this method we have made calculations for a number of problems which admit an exact solution. The agreement of the results is good without exception. In Figs. 1 and 2 we give the results of comparing the approximate solution obtained by our method and the exact solution for two problems.

In the first case we solved a problem with the following initial and boundary values:

$$u|_{t=0} = 0, \quad P|_{t=0} = 0, \quad E|_{t=0} = 0, \quad \rho|_{t=0} = 1 \quad \left(\rho = \frac{1}{V}\right), \quad P|_{x=0} = 1, \quad u|_{x=50} = 0.$$

In the second case the initial and boundary values were given in the following way: $u|_{t=0} = 0$, $P|_{t=0} = 0$, $E|_{t=0} = 0$, $P|_{x=0} = 1$, $u|_{x=50} = 0$,

$$\rho|_{t=0} = \begin{cases} 1 & \text{for } 0 \leqslant x \leqslant 29.5, \\ 0.5 & \text{for } 29.5 < x \leqslant 50. \end{cases}$$

In both cases the calculations were done with $k = 0.5$.

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