Analytical Solution of the Problem of a Shock Wave in the Collapsing Gas in Lagrangian Coordinates

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Abstract. It is proposed the exact solution of the problem of a convergent shock wave and gas dynamic compression in a spherical vessel with an impermeable wall in Lagrangian coordinates. At the initial time the speed of cold ideal gas is equal to zero, and a negative velocity is set on boundary of the sphere. When $t > t_0$ the shock wave spreads from this point into the gas. The boundary of the sphere will move under the certain law correlated with the motion of the shock wave. The trajectories of the gas particles in Lagrangian coordinates are straight lines. The equations determining the structure of the gas flow between the shock front and gas border have been found as a function of time and Lagrangian coordinate. The dependence of the entropy on the velocity of the shock wave has been found too. For Lagrangian coordinates the problem is first solved. It is fundamentally different from previously known formulations of the problem of the self-convergence of the self-similar shock wave to the center of symmetry and its reflection from the center, which was built up for the infinite area in Euler coordinates.

The border of the gas sphere with the mass M_0 and initial parameters for the gas $\rho_0 = \text{const}$, $U_0 = 0$, $P_0 = 0$, $E_0 = 0$, where ρ – density, U – velocity, P – pressure, E – specific internal energy, with jump starts moving with a negative velocity at the moment $t = t_0$. When $t > t_0$ the boundary trajectory changes in variables r, t, but it is a vertical line in variables M, t. Generally speaking, all the trajectories of the particles are vertical lines in the variables M, t. Along each of them retained the value of entropy, which appeared on the shock wave spreading from the border to the gas. Conservation laws on the shock wave at $U_0 = 0$, $P_0 = 0$, $E_0 = 0$ are of the form [1]

$$\rho_w \left(D - U_w\right) - \rho_0 D = 0, \tag{1}$$

$$\rho_0 D U_w - P_w = 0, \tag{2}$$

$$\rho_0 D\left(E_w + \frac{1}{2}U_w^2\right) - P_w U_w = 0. \tag{3}$$

The index "*w*" indicats values on the shock wave, D - the shock wave velocity. Equations (1) - (3) closed by the equation of state

$$P = \left(\gamma - 1\right)\rho E\ . \tag{4}$$

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In contrast to [2, 3], solution of the problem will be constructed in Lagrangian coordinates. The Lagrangian coordinate M_w of the shock wave in the case of a spherically symmetric flow associated by the equation with its coordinate Euler r_w

$$M_w = \frac{4}{3}\pi\rho_0 r_w^3. \tag{5}$$

The shock wave velocity W in Lagrangian coordinates and velocity D in Euler coordinates associated by the equation

$$W = \left(3M_w\right)^{2/3}\left(4\pi\rho_0\right)^{1/3} D. \tag{6}$$

Expressing in (6) D through W and M_w and substituting in (1) - (3), we obtain the conditions on the shock wave, containing W and M_w

$$\left(\frac{1}{\rho_w}-\frac{1}{\rho_0}\right)W + \left(4\pi\right)^{1/3}\left(\frac{3M_w}{\rho_0}\right)^{2/3} U_w = 0, \quad U_w W - \left(4\pi\right)^{1/3}\left(\frac{3M_w}{\rho_0}\right)^{2/3} P_w = 0, \tag{7}$$

$$\left(E_w + 0,5U_w^2\right)W - \left(4\pi\right)^{1/3}\left(\frac{3M_w}{\rho_0}\right)^{2/3} P_w U_w = 0. \tag{8}$$

From (4), (7), (8) ensue expression for ρ_w, U_w and P_w

$$\rho_w = \frac{\gamma+1}{\gamma-1}\rho_0, \quad U_w = \frac{2W}{\left(\gamma+1\right)\left(4\pi\rho_0\right)^{1/3}\left(3M_w\right)^{2/3}}, \quad P_w = \frac{2\rho_0^{1/3} W^2}{\left(\gamma+1\right)\left(4\pi\right)^{2/3}\left(3M_w\right)^{4/3}}. \tag{9}$$

Substituting ρ_w and P_w in the equation of the state of the ideal gas in the form $P = F\left(S\right)\rho^\gamma$, we obtain

$$F_w = \frac{2}{\gamma+1}\left(\frac{\gamma-1}{\gamma+1}\right)^\gamma \frac{W^2\rho_0^{(1-3\gamma)/3}}{\left(4\pi\right)^{2/3}\left(3M_w\right)^{4/3}}. \tag{10}$$

At the point t_0, M_0 $W = W_1, M_w = M_0$, $P_w = P_1$, $F_w = F_1$, $\rho_w = \rho_1$, $U_w = U_1$, then the expressions (9) and (10) take the form

$$U_w = U_1\left(\frac{W}{W_1}\right)\left(\frac{M_0}{M_w}\right)^{2/3}, \quad P_w = P_1\left(\frac{W}{W_1}\right)^2\left(\frac{M_0}{M_w}\right)^{4/3}, \quad F_w = F_1\left(\frac{W}{W_1}\right)^2\left(\frac{M_0}{M_w}\right)^{4/3}. \tag{11}$$

We define the trajectory of the shock wave in the form

$$M_w = M_0\left(\frac{t_f - t}{t_f - t_0}\right)^n, \tag{12}$$

where t_f - focusing time. The shock wave velocity in Lagrangian coordinates depends on the time

$$W = W_1\left(\frac{t_f - t}{t_f - t_0}\right)^{n-1}, \tag{13}$$

where $W_0 = -M_0\, n / \left(t_f - t_0\right)$. Excluding the functions of the time in (12) and (13), we obtain

$$\frac{W}{W_0} = \left(\frac{M_w}{M_0}\right)^{(n-1)/n}. \tag{14}$$

With the help of (14) from (10) the dependence F_w from M_w is obtained. Because both F_w and M_w along the trajectory of each particle is constant, the dependence of the entropy on the mass between the shock wave and the boundary of the gas has the form

$$F = F_1\left(\frac{M}{M_0}\right)^{2(n-3)/3n}. \tag{15}$$

Parameters of the adiabatic flow between the shock wave and boundary of the gas are determined by the equations of the trajectory, the conservation of mass and motion

$$\left(\frac{\partial r}{\partial t}\right)_M - U = 0, \qquad \left(\frac{\partial \rho}{\partial t}\right)_M + 4\pi\rho^2 \frac{\partial\left(r^2 U\right)}{\partial M} = 0\,, \tag{16}$$

$$\left(\frac{\partial U}{\partial t}\right)_M + 4\pi r^2 \frac{\partial\left(F\rho^\gamma\right)}{\partial M} = 0\,. \tag{17}$$

These equations contain three desired functions $r,\ \rho$ and U. The value F is determined on the shock wave and depends only on M (15).

Let us pass in (16), (17) to new desired functions

$$R = r^3, \quad C = r^2 U. \tag{18}$$

After the pass to the functions R and C the equations (16) - (17) take the form

$$\left(\frac{\partial R}{\partial t}\right)_M - 3C = 0, \qquad \left(\frac{\partial \rho}{\partial t}\right)_M + 4\pi\rho^2 \frac{\partial C}{\partial M} = 0\,, \tag{19}$$

$$\left(\frac{\partial C}{\partial t}\right)_M + 4\pi R^{\frac{4}{3}} \frac{\partial\left(F\rho^\gamma\right)}{\partial M} - 2C^2 R^{-1} = 0\,. \tag{20}$$

From (5), (9) and (14) follow the dependence of R_w and C_w on M_w

$$R_w = R_0 \frac{M_w}{M_0}, \quad C_w = C_0 \left(\frac{M_w}{M_0}\right)^{(n-1)/n}. \tag{21}$$

The equations (19) and (20) are essential for finding R, C and ρ in the area of the integration $M_w \le M \le M_0$, $t_0 \le t \le t_f$.

Let us proceed from the variables t, M to variables $t, \xi(t, M)$. With the help of the equations for the derivatives

$$\left(\frac{\partial}{\partial t}\right)_M = \left(\frac{\partial}{\partial t}\right)_\xi + \left(\frac{\partial}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial t}\right)_M, \qquad \left(\frac{\partial}{\partial M}\right)_t = \left(\frac{\partial}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial M}\right)_t$$

we transform the equations (19)–(20)

$$\left(\frac{\partial R}{\partial t}\right)_\xi + \left(\frac{\partial R}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial t}\right)_M - 3C = 0\,, \tag{22}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_\xi + \left(\frac{\partial \rho}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial t}\right)_M + 4\pi\rho^2 \left(\frac{\partial C}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial M}\right)_t = 0\,, \tag{23}$$

$$\left(\frac{\partial C}{\partial t}\right)_\xi + \left(\frac{\partial C}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial t}\right)_M - \frac{2C^2}{R} + 4\pi R^{\frac{4}{3}} \left[\rho^\gamma \left(\frac{\partial F}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial M}\right)_t + \gamma F \rho^{\gamma-1} \left(\frac{\partial \rho}{\partial \xi}\right)_t \left(\frac{\partial \xi}{\partial M}\right)_t\right] = 0. \tag{24}$$

We define the dependence of $\xi(t, M)$ in the form

$$\xi = \frac{M}{M_0}\left(\frac{t_f - t}{t_f - t_0}\right)^{-n}. \tag{25}$$

It follows from (12) and (25) that on the shock wave at $M = M_w$ will be $\xi = 1$.

To separate the variables representing R, ρ and C in the form of products of functions of the time and the function of ξ

$$R = \alpha_R(t)\, T(\xi) \qquad \rho = \alpha_\rho(t)\, \delta(\xi) \qquad C = \alpha_C(t)\, Z(\xi)\,. \tag{26}$$

Since $\xi = 1$ on the shock wave, the values $T_w(1)$, $\delta_w(1)$, $Z_w(1)$ must be constant. The dependence $R_w(\xi, t)$ is obtained at $\xi = 1$ from (21) and (25)

$$R_w = R_0 \cdot \left(\frac{t_f - t}{t_f - t_0}\right)^n. \tag{27}$$

To eliminate the arbitrary choice in the separation of functions R, ρ, C in (26) we require that $T(\xi)$, $\delta(\xi)$, $Z(\xi)$ would be dimensionless. Then from (26) and (27) we go to

$$T_w = 1, \quad \alpha_R(t) = R_0 \left(\frac{t_f - t}{t_f - t_0} \right)^n . \tag{28}$$

Similarly for ρ and C we obtain the relations

$$\delta_w = \frac{\gamma+1}{\gamma-1}, \quad Z_w = 1, \quad \alpha_\rho = \rho_0, \quad \alpha_C(t) = C_0 \left(\frac{t_f - t}{t_f - t_0} \right)^{n-1} . \tag{29}$$

By substituting (26) - (29) (22) - (24), we obtain three equations for T, δ and Z

$$\xi T' = A_1 , \tag{30}$$

$$B_1 Z' - \xi \delta' = 0 , \tag{31}$$

$$-\xi Z' + C_1 \gamma \xi \delta' = C_2 , \tag{32}$$

where the mark indicates differentiation of ξ,

$$A_1 = T - \frac{3ZM_0C_0}{W_0R_0}, \quad B_1 = \frac{4\pi\rho_0\delta^2 C_0}{W_0}, \quad C_1 = 4\pi\rho_0^\gamma F_0 T^{4/3}\delta^{\gamma-1}\xi^{-(n+6)/3n} R_0^{4/3} / C_0 W_0 ,$$

$$C_2 = \frac{3Z^2 M_0 C_0}{TR_0W_0} - \frac{(n-1)Z}{n} - C_1 \frac{2(n-3)\delta}{3n} .$$

Using (9) and the relations following from (5), (6) and (18)

$$M_0 = \frac{4}{3}\pi\rho_0 r_0^3, \quad W_0 = 4\pi\rho_0 r_0^2 D_0, \quad C_0 = \frac{2}{\gamma+1} r_0^2 D_0, \quad R_0 = r_0^3$$

we will simplify the relations for the coefficients of the equations (29) - (31).

$$\begin{gathered} A_1 = T - \frac{2Z}{\gamma+1}, \quad B_1 = \frac{2}{\gamma+1}\delta^2, \quad C_1 = \left(\frac{\gamma-1}{\gamma+1} \right)^\gamma T^{4/3}\delta^{\gamma-1}\xi^{-(n+6)/3n}, \\ C_2 = \frac{4Z^2}{3(\gamma+1)T} - \frac{(n-1)Z}{n} - C_1 \frac{2(n-3)\delta}{3n} . \end{gathered} \tag{33}$$

The equations (30), (31) are the system of linear inhomogeneous equations regarding to the T', δ', Z'. The determinant of the system is the following

$$\Delta = B_1 C_1 \gamma \xi - \xi^2 .$$

If $\Delta \neq 0$, the solution of the system (29) - (31) has the form

$$T' = \frac{A_1}{\xi}, \quad \delta' = \frac{B_1 C_2}{\Delta}, \quad Z^1 = \frac{\xi C_2}{\Delta} . \tag{34}$$

Integrating the system of the equations (29) - (31) begins at the point $\xi = 1$ (on the shock wave), where $T_w = 1,\ Z_w = 1,\ \delta_w = \frac{\gamma+1}{\gamma-1}$ and, consequently,

$$\begin{gathered} A_1 = \frac{\gamma-1}{\gamma+1}, \ B_1 = \frac{2(\gamma+1)}{(\gamma-1)^2}, \ C_1 = \frac{\gamma-1}{\gamma+1}, \\ C_2 = \frac{9(\gamma+1) - n(5\gamma+1)}{3n(\gamma+1)}, \ \Delta(1) = \frac{\gamma+1}{\gamma-1}. \end{gathered}$$

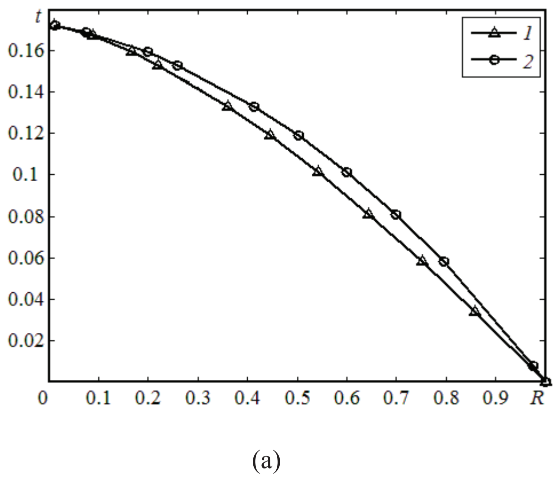
Calculations show that there exist interval values of n such that the determinant does not vanish. At some value of n_* the determinant vanishes. In this case, there is a solution, if C_2 is also vanishes. The values n_* corresponding to the values γ are given in the Table 1. In this table shows the values ξ_*, in which simultaneously $\Delta(\xi_*)=0$, $C_2(\xi_*)=0$. The value n_* is separates the solution of the problem into two types

TABLE 1. The values n_* and ξ_* corresponding to the values γ.

γ	n_*	ξ_*
1.1	2.387895	7.915
1.2	2.271414	5.695
4/3	2.183052	4.555
1.4	2.150000	4.285
5/3	2.065128	3.505

In case, when $0 < n < n_*$ there is a collapse of the gas sphere - its volume approaches to zero.

The trajectories of the shock wave and the border of the gas sphere, profiles of velocities, pressures and densities are given in Fig. 1 - 3.



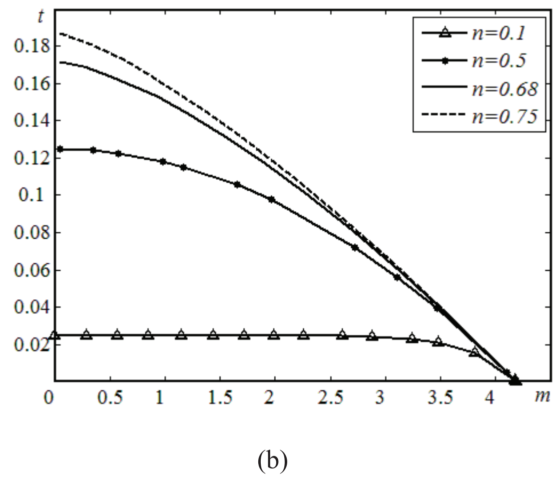
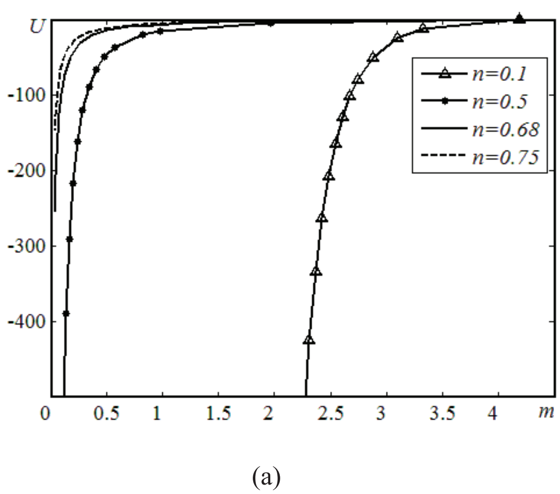


FIGURE 1. The trajectories of the shock wave and the border of the gas sphere for $\gamma = 5/3$ and $n = 0,68$ (a). The trajectories of the shock waves for $\gamma = 5/3$ and different values $0 < n < n_*$ in Lagrangian coordinates (b).



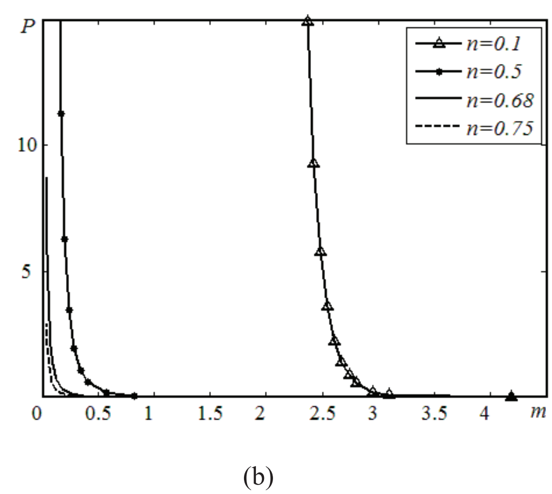
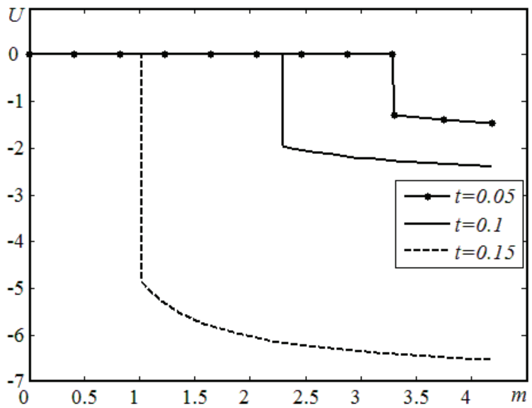
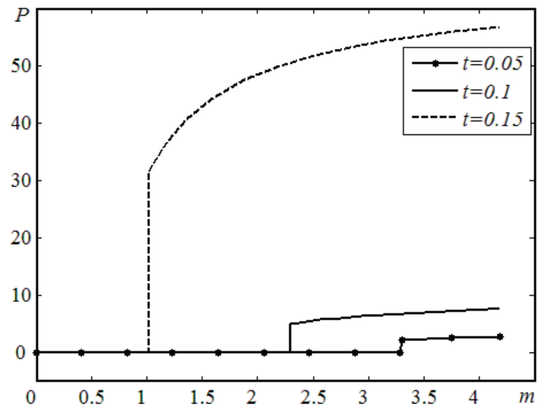


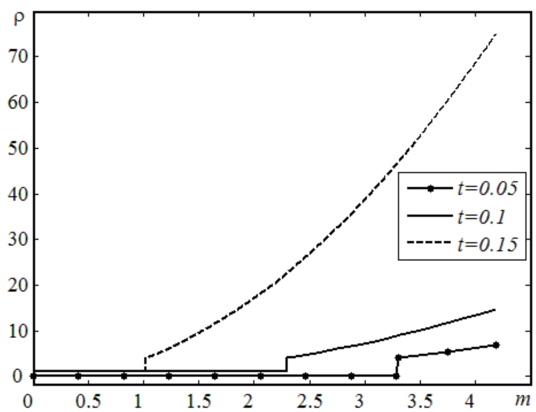
FIGURE 2. Profiles of velocities (a), pressures (b) on the shock wave for different values $0 < n < n_*$ and $\gamma = 5/3$ in Lagrangian coordinates.



(a)



(b)



(c)

FIGURE 3. Profiles of velocities (a), pressures (b) and densities (c) for $\gamma = 5/3$ and $n = 0,68$ for different time points.

In the area $n > n_*$ the determinant vanishes for some value ξ_n, which is depends on n. But at this point $C_2(\xi_n)$ does not vanish. Thus, the solution exists in the area $1 \le \xi < \xi_n$. On the border of the gas sphere at $M=M_0$ the value of ξ_n is reached at the moment

$$t_n = t_f - \left(t_f - t_0\right)\xi_n^{1/n}. \tag{35}$$

This is follows from the equation (25). From the point M_0, t_n comes out the line, on which $\xi=\xi_n$. The equation of line is obtained from (25)

$$M_n = M_0 \xi_n \left(\frac{t_f - t}{t_f - t_0}\right)^n. \tag{36}$$

This line is focuses simultaneously with the shock wave, because $M_n=0$ at $t=t_f$. In the area between (36) and the shock wave (12), for each $n>n_*$ the single solution is exists.

The analytical solution of the problem of a converging shock wave in the collapsing gas has been constructed in Lagrangian coordinates with arbitrary parameter n, which determines the convergence of the shock wave.

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