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HYDROGASDYNAMICS IN TECHNOLOGICAL PROCESSES

MODIFIED MATHEMATICAL MODEL OF A "FROZEN" GAS SUSPENSION

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An analysis has been made of the invariance of a widely used mathematical model of a "frozen" gas suspension with respect to the Galilean transformation. It has been shown that the equations of this model are not invariant with respect to the Galilean transformation, which results in a fictitious source term in the energy equation. Additional entropy increase leads to a violation of the second law of thermodynamics. In this work, the authors have proposed a modification of the "frozen"-gas-suspension model that leads to its invariance with respect to the Galilean transformation. Calculations have been performed and comparisons with an invariant model have been made.

Keywords: *mathematical model, invariance, multicomponent mixture.*

Introduction. The wide use of explosive processes in a number of fields of modern technology is closely connected with the solution of issues of ensuring safety and protection of engineering structures and technological equipment against the action of shock waves (SWs). In this connection, studying the problem of localization of mechanical effects of explosions and of SW attenuation using mathematical modeling of these physical processes is of great applied importance [1]. Therefore, there is a particular acuteness of the problem of both the development of mathematical models of multicomponent heterogeneous media [2], which are based on the hypothesis of interpenetrating interacting continua [3], and an analysis of the already existing mathematical models [4, 5]. The development of modern computation engineering has made it possible to significantly increase the sophistication of mathematical models of physical processes used in science and technology. Due to this, the status of mathematical modeling as a source of information on the processes has risen. Moreover, for fast processes, mathematical modeling is often the only means for preliminary study of phenomena [6, 7].

To verify calculations one uses, on the one hand, the available experimental data, and on the other, mathematical models in analyzing the performed measurements. It is of first importance that conditions for performing calculations and experimental conditions coincide, and that the mathematical model is invariant with respect to the Galilean transformation.

In [4, 5], we analyzed a mathematical model of a "frozen" gas suspension that is actively used in analyzing the attenuation of SWs in heterogeneous media. It has turned out that the invariance of the equations from [6] with respect to the Galilean transformation gives rise to an additional energy source due to the motion of the coordinate system. This energy source has no physical nature and leads to a violation of the second law of thermodynamics.

In the present work, we have proposed a modification of the mathematical "frozen"-gas- suspension model from [8], calculated the attenuation of an SW in a system of shields, and made a comparison with the results from [8].

Formulation of the Problem. When the problem of attenuation of an SW using an aerosuspension is solved, it is assumed that particles of the solid component are quiescent and incompressible as a rule. This means that a gas-filled undeformable lattice (grill) is actually considered instead of the gas suspension. Solid particles simulate lattice sites, and bonds between these sites exert no influence on gasdynamic flow, i.e., when the attenuation of SWSs is studied, use is made of the "frozen"-gas-suspension model presented in [5, 6]. Since the particles are quiescent and incompressible, their volume concentration α and hence the volume concentration of the gas $1 - \alpha$ are constant.

A two-component medium was considered in [8] to determine the effect of attenuation of an SW using the grill. The behavior of the gas (quantities without subscripts) in the mixture is described by the equations

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$$\frac{d\rho}{dt} + \rho \frac{\partial u}{\partial x} = 0 , \tag{1}$$

$$(1 - \alpha)\rho \frac{du}{dt} + \frac{\partial P}{\partial x} = -F , \tag{2}$$

$$\rho \frac{dE}{dt} + \frac{\partial Pu}{\partial x} = - \frac{Q}{(1 - \alpha)} . \tag{3}$$

System (1)–(3) is supplemented with the expressions for the total energy

$$E = e + \frac{1}{2} u^2 \tag{4}$$

and by the equation of state

$$P = (\gamma - 1)\rho e \text{ at } \gamma = 1.4 . \tag{5}$$

The parameters of the grill (subscript g) are described by the following expressions:

$$\rho_g = \text{const}, \quad u_g = 0, \quad P_g = P, \quad \alpha\rho_g \frac{\partial e_g}{\partial t} = Q . \tag{6}$$

The substantive derivative is determined in the following manner:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} .$$

The initial data of the problem are formulated analogously to [8]. At $t = 0$, the air–iron mixture is in the region $x_1 \le x \le x_2$ under the normal conditions $P_0 = 0.1$ MPa, $\rho_0 = 1.21$ kg/m^3, $\rho_g = 7800$ kg/m^3, $u_g = 0$, and $T = 300$ K.

The boundary $x = x_1$ is the contact between a pure gas and a mixture with $\alpha > 0$. At the instant $t = 0$, this contact boundary is reached by the SW on whose front we have the pressure $P_+ = 2$ MPa. All the remaining quantities on the SW front are determined from the Hugoniot conditions: $\rho_+ = 5.631$ kg/m^3, $e_+ = 0.88795$ MJ/kg, $u_+ = 1.110333$ km/s, and $E_+ = 0.70516$ MJ/kg. Thus, on the $x = x_1$ surface, there has been formed an arbitrary discontinuity (gas on the left and the mixture on the right). At $t > 0$, the discontinuity disintegrates to form a wave that has penetrated into the mixture and a reflected SW. In the region $0 \le x \le x_2$, at the instant $t = 0$, we prescribe a rarefaction wave with a linear velocity profile:

$$u = u_+ \left(\frac{x}{x_1} \right) .$$

The remaining quantities are determined by the equations

$$C = C_+ - \frac{\gamma - 1}{2} u_+ \left(1 - \frac{x}{x_1} \right) ,$$

$$\rho = \rho_+ \left(\frac{C}{C_+} \right)^{\frac{2}{\gamma-1}} , \quad e = e_+ \left(\frac{C}{C_+} \right)^2 , \quad P = (\gamma - 1) \rho e .$$

At $t \ge 0$, we specify the condition $u = 0$ on the right-hand boundary $x = x_2$ and the condition of free flow of the gas on the left-hand boundary at $x = 0$.

Analysis of the Model of [8]. Following [2], we write the entropy-production equation in model (1)–(3). For this purpose, we eliminate the kinetic energy from the energy equation (3) using the equation of motion (2) and replace the derivative $\frac{\partial u}{\partial x}$ by the derivative $\frac{1}{\rho} \frac{\partial \rho}{\partial t}$ using (1). This yields

$$\frac{\partial e}{\partial t} + P\frac{dV}{dt} = \frac{1}{\rho(1-\alpha)}\left(uF - Q + \alpha u\frac{\partial P}{\partial x}\right), \tag{7}$$

where $V = 1/\rho$.

We compare Eq. (7), which is a consequence of Eqs. (1)–(3), and the expression for the specific internal energy of the gas

$$\frac{de}{dt} + P\frac{dV}{dt} = T\frac{dS}{dt}, \tag{8}$$

where T determines the temperature, and S determines the entropy. It follows from (7) and (8) that the entropy of the gas in its motion between grill particles varies according to the equations

$$T\frac{dS}{dt} = -\frac{Q}{\rho(1-\alpha)} + \frac{u}{\rho(1-\alpha)}\left(F + \alpha\frac{\partial P}{\partial x}\right). \tag{9}$$

If there is no grill, the quantities α, F, and Q are zero, and the entropy remains constant along the trajectories of the gas particles. However, if $\alpha > 0$, the gas is additionally heated because of the work $\frac{u}{\rho(1-\alpha)}\left(F + \alpha\frac{\partial P}{\partial x}\right)$, with this term being such that $\alpha \to 1$ $\frac{dS}{dt} \to \infty$, i.e., the gas experiences a thermal explosion. Also, it follows from Eq. (9) that the exchanges of momenta and thermal energy in model (1)–(3) are absolutely irreversible.

Let us analyze the invariance of the system of Eqs. (1)–(3) with respect to the Galilean transformation. For this purpose, we change to a new coordinate system that moves with a constant velocity D. The velocity and the coordinate will vary thus:

$$u_{\mathrm{n}} = u + D, \quad x_{\mathrm{n}} = x + Dt.$$

The derivatives with respect to the coordinate and time are determined as follows:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_{\mathrm{n}}}, \quad \left(\frac{\partial}{\partial t}\right) = \left(\frac{\partial}{\partial t}\right)_{\mathrm{n}} + \left(\frac{\partial}{\partial x_{\mathrm{n}}}\right)D.$$

After the changeover to the moving coordinate system, we will drop the "n" subscript. The functions F and Q remain constant on the changeover to the new coordinate system.

It is easily checked that Eqs. (1) and (2) for the gas are invariant with respect to the Galilean transformation, and Eq. (2) for the grill is invariant and must be replaced by the equation

$$\frac{d_{\mathrm{g}}u_{\mathrm{g}}}{dt} = 0. \tag{10}$$

Next, we consider (3) for the gas. In the new coordinate system moving with a constant velocity D, Eq. (3) is of the form

$$\rho\frac{dE}{dt} + \frac{\partial Pu}{\partial x} + \frac{Q}{(1-\alpha)} = \omega, \tag{11}$$

where

$$\omega = \frac{u-D}{(1-\alpha)}\left(F + \alpha\frac{\partial P}{\partial x}\right). \tag{12}$$

Transforming (11) analogously to a fixed coordinate system, we obtain the entropy-production equation

$$T\frac{dS}{dt} = -\frac{Q}{\rho(1-\alpha)} + \frac{u-D}{\rho(1-\alpha)}\left(F + \alpha\frac{\partial P}{\partial x}\right). \tag{13}$$

It is seen from a comparison of Eqs. (9) and (13) that the second terms on the right-hand sides depend on the selection of a coordinate system, i.e., on D. In other words, the energy equation (3) is invariant with respect to the Galilean transformation. If we divide the entropy into two parts

$$S = S_{\text{ph}} + S_{\text{G}} ,$$

where S_{ph} is governed by the model′s "physics" and S_{G} is determined by the Galilean invariance, we obtain the additional entropy-production equation

$$T \frac{dS_{\text{G}}}{dt} = \frac{u - D}{\rho(1 - \alpha)} \left(F + \alpha \frac{\partial P}{\partial x} \right) , \tag{14}$$

which is only due to the fact that the authors of the model from [8] have disregarded the fundamental principles of mechanics.

Modification of the Model of [8]. Note that since we have α = const in the mixture, according to [2], the factors $(1 - \alpha)$ before the derivative of velocity with respect to time in (2) and $1/(1 - \alpha)$ before the heat-exchange function in (3) must be absent. Furthermore, the intercomponent-interaction force F acts in the equation of motion (2), whereas the work of this force is absent from the energy equation (3), which is one reason for the invariance of the energy equation (3) with respect to the Galilean transformation. Taking account of these notes, we write the equations of motion and energy of the gas in the form

$$\rho \frac{du}{dt} + \frac{\partial P}{\partial x} = -F , \tag{15}$$

$$\rho \frac{dE}{dt} + \frac{\partial Pu}{\partial x} = -Q - Fu . \tag{16}$$

After all the necessary transformations on changing to the coordinate system moving with a constant velocity D, we obtain that the model of the mixture, in which the behavior of the gas is determined by Eqs. (1), (15), and (16), is invariant with respect to the Galilean transformation, and its entropy-production equation is of the form

$$T \frac{dS}{dt} = -\frac{Q}{\rho} . \tag{17}$$

Equation (17) is consistent with the second law of thermodynamics. Clearly, the entropy production of the gas phase is only determined by interphase heat exchange.

Unfortunately, the principle of invariance with respect to the Galilean transformation is not observed in a number of multicomponent-medium models reported in scientific journals. Such models are incapable of predicting results of physical processes for whose modeling they are intended.

Comparison of Two Models. We investigate the mechanical action of SWs on the obstacle wall as a function of the concentration of a condensed material in grills and the dimensions of their sites. An analysis of the mechanical action of an SW on the obstacle wall will be performed using the "frozen"-gas-suspension model from [8] and an updated model in which the behavior of the gas is determined by Eqs. (1), (4)–(6), (15), and (16). Expressions for the intensity of the "external" force and thermal interaction of the gas with the grill are prescribed with account of the constraint of the sites [8] in the following manner:

$$F = 0.75 C_{\text{d}} \alpha \rho \, |u| \, u d^{-1} , \quad C_{\text{d}} = C_{\text{d}}^0(\text{Re}) \Psi(\text{M}) \varphi(\alpha) ,$$

$$C_{\text{d}}^0 = 24/\text{Re} + 4/\sqrt{\text{Re}} + 0.4, \quad \varphi(\alpha) = (1 - \alpha)^{-n}, \quad n = 1 ,$$

$$\Psi(\text{M}) = \left[1 + \exp\left(-0.427/\text{M}^{4.63}\right) \right], \quad \text{M} = \frac{u}{c} ,$$

$$c = \sqrt{\gamma p/\rho}, \quad \text{Re} = \frac{\rho |u| \, d}{\mu} , \quad Q = 6\alpha\lambda \text{Nu} \left(\frac{e}{c_v} - \frac{e_g}{c_{vg}} \right) d^{-2} ,$$

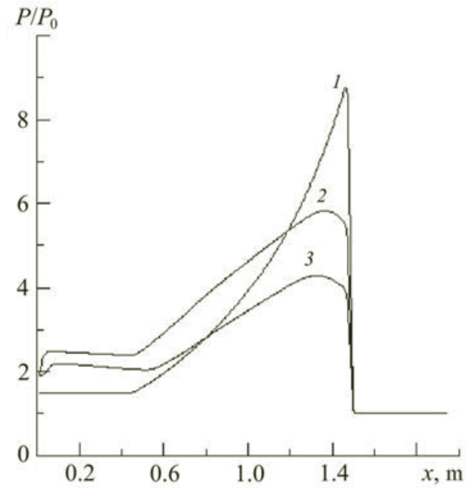

Fig. 1. Dimensionless pressure vs. coordinate at the instant of the SW reaching the point $x = 1.5$ m: 1) calculation in pure gas at $\alpha = 0$, 2) calculation from the invariant model at $\alpha = 0.003$, and 3) calculation from the modified model at $\alpha = 0.003$.

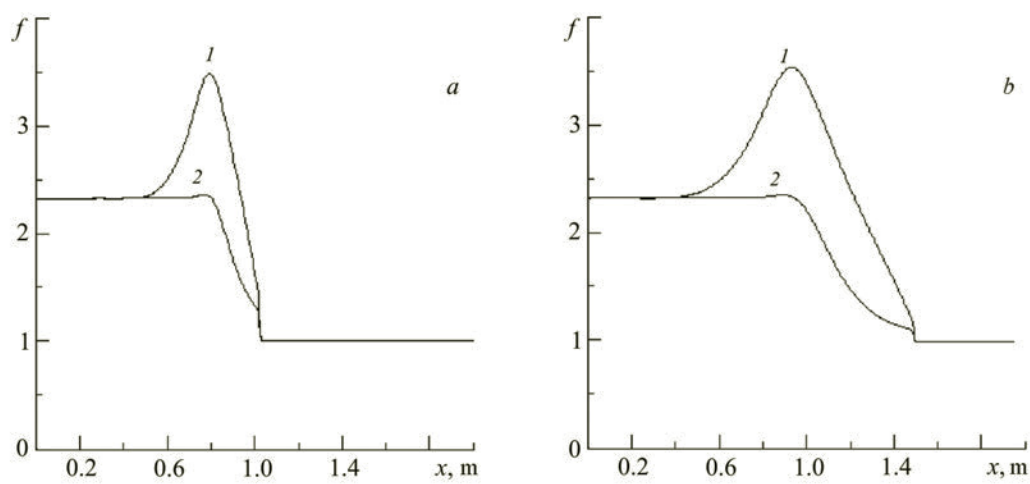

Fig. 2. Dimensionless entropy function $f = \dfrac{P}{P_0}\left(\dfrac{\rho_0}{\rho}\right)^{\gamma}$ vs. coordinate x at the instant of the SW reaching the point $x = 1$ (a) and $x = 1.5$ m (b): 1) calculation from the noninvariant model [8] at $\alpha = 0.003$ and 2) calculation from the modified model at $\alpha = 0.003$.

$$\mathrm{Nu} = 2\exp\left(-M\right) + 0.459\ \mathrm{Re}^{0.55}\ \mathrm{Pr}^{0.33}, \quad \mathrm{Pr} = \frac{c_p\mu}{\lambda}.$$

In the above dependences, C_{d} is the coefficient of aerodynamic drag of a sphere, which has been written with account of the flow compressibility (function $\Psi(\mathrm{M})$) and of the particle constraint (function $\varphi(\alpha)$), and C_{d}^{0} is the coefficient of aerodynamic drag of a particle in an infinite incompressible gas flow. The formulated problem was solved by the large-particle method [9] according to the algorithm of [10]. Calculations were done for the system "air–iron." The selected coordinates of the boundaries were as follows [8]: $x_0 = 0$, $x_1 = 0.45$ m, and $x_2 = 1.95$ m.

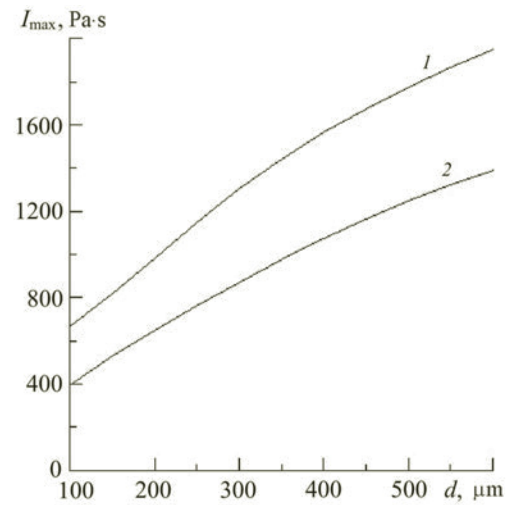

Fig. 3. Excess momentum I_{max} transferred to the wall vs. particle diameter: 1) calculation from the noninvariant model [8] at α = 0.003 and 2) calculation from the modified model at α = 0.003.

Figure 1 gives the pressure profiles at the instant of time when the SW is at the point $x = 1.5$ m: 1) pressure distribution in the absence of a grill; 2) calculation for the mathematical "frozen"-gas-suspension model of [8] and 3) for the modified mathematical model. It is clear from a comparison of plots 2 and 3 that the addition of the work of intercomponent-interaction forces to the energy equation of the gas phase leads to a much stronger SW attenuation than that within the framework of the mathematical "frozen"-gas-suspension model from [8]. Figure 2 shows the profiles of a dimensionless entropy function when the SW is at the points $x = 1.0$ (a) and $x = 1.5$ (b) respectively. It is seen in the figures that the modified mathematical "frozen"-gas-suspension model describes isentropic flow of the gas phase behind the SW (curves 2) indeed, unlike the "frozen"-gas- suspension model of [8] (curves 1). Since the SW intensity in the calculations with the model of [8] drops much more slowly than within the framework of the modified mathematical model, the momentum transferred to the wall will be larger. This is confirmed in Fig. 3 where the profiles of the momentum transferred to the wall are plotted versus the particle diameter.

CONCLUSIONS

1. The analysis performed in the present work has shown that the model of the mixture proposed in [8] and described by Eqs. (1)–(6) is not invariant with respect to the Galilean transformation; consequently, calculations by the large-particle method that employ this model are not reliable.

2. The modified mathematical "frozen"-gas-suspension model ((1), (4)–(6), (15), and (16)) proposed in the present work is invariant with respect to the Galilean transformation.

3. The results of the interaction of the SW with the grill, obtained from the above material, differ significantly.

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NOTATION

c, velocity of sound, km/s; c_p, specific heat at constant pressure, J/(kg·K); c_v, specific heat at constant volume, J/(kg·K); C_d, aerodynamic-drag coefficient; d, particle diameter, μm; D, velocity of the coordinate system, km/s; E and e, specific total and internal energies, MJ/kg; F, intercomponent-interaction force, N; M, Mach number; Nu, Nusselt number; P, pressure, MPa; Pr, Prandtl number; Q, intercomponent heat exchange, W/m^3; Re, Reynolds number; S, entropy, J/K; t, time, s; u, velocity, km/s; V, specific volume, m^3/kg; x, coordinate, m; α, volume concentration of the solid component; γ, adiabatic exponent; λ, thermal conductivity, W/(m·K); ρ, density, kg/m^3; Ψ, dependence of the aerodynamic-drag coefficient on the Mach number. Subscripts: g, grill; n, index of the parameters in the new coordinate system.

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