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MECHANICS

Sound Velocity in a Multicomponent Mixture

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Components of a multicomponent mixture (MCM) can be in any aggregate state. The properties of the mixture and the character of the interaction between the components depend strongly on the state of each component and on the interaction character in the MCM. The overwhelming majority of substances are mixtures to some degree. One promising method for nondestructive monitoring of the medium is the pulsed acoustic diagnostics, which is based on the measurement of the evolution of sound signals in the medium. In these methods, the sound velocity plays an essential role. When determining the sound velocity in a mixture, various authors make various simplifying assumptions. As a result, various formulas relating the propagation velocity of small perturbations in the mixture with the velocities of sound for the components can be derived. We analyze various equations for the sound velocity in a mixture and suggest and substantiate the formula describing the dependence of the sound velocity in a mixture on the sound velocities and bulk concentrations of components adequately.

Sound velocity C in an ideal uniform medium is determined by the Laplace formula

$$C^2 = \left(\frac{\partial P}{\partial \rho}\right)_S, \tag{1}$$

where P is the pressure, ρ is the density, and S is the entropy. The consequence of applying formula (1) to derive the sound velocity in the binary mixture [1–3] in a certain system of simplifying hypotheses and $N = 2$ is the formula

$$\frac{1}{\rho C^2} = \sum_{i=1}^{N} \frac{\alpha_i}{\rho_i C_i^2}, \tag{2}$$

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where ρ_i, C_i, α_i are the density, the sound velocity, and the bulk concentration of the ith component; i is the component number; and ρ, C are the density and the sound velocity for the mixture. The uniqueness of formula (2) is that the α-dependence of C is nonmonotonic. It reaches the maximal value

$$C_m = \frac{2C_1C_2\sqrt{\rho_1\rho_2(\rho_2-\rho_1)(\rho_2C_2^2-\rho_1C_1^2)}}{\rho_2^2C_2^2-\rho_1^2C_1^2}$$

at

$$\alpha_m = \frac{C_1^2(\rho_2-\rho_1)^2+\rho_2^2(C_2^2-C_1^2)}{2(\rho_2-\rho_1)(\rho_2C_2^2-\rho_1C_1^2)}.$$

It is seen from [1–4] that formula (2) has been repeated in various monographs for half a century.

Let us show that we can also derive other dependences $C(C_i, \alpha_i)$ from (1) with the same simplifying hypotheses. The parameters of the mixture are related with the corresponding parameters of components by instantaneous conservation laws and their consequences:

$$P = \sum_{i=1}^{N} \alpha_i P_i, \quad S = \sum_{i=1}^{N} \eta_i S_i, \tag{3}$$

$$V = \sum_{i=1}^{N} \eta_i V_i, \tag{4}$$

$$\rho = \sum_{i=1}^{N} \alpha_i \rho_i, \tag{5}$$

$$u = \sum_{i=1}^{N} \eta_i u_i, \tag{6}$$

where P_i, S_i, u_i, and η_i are the pressure, the entropy, the velocity, and the weight concentration of the ith component; and N is the number of components. Concentrations α_i and η_i satisfy the equations

$$\sum_{i=1}^{N}\alpha_i = 1,\quad \sum_{i=1}^{N}\eta_i = 1,\quad \eta_i\rho = \alpha_i\rho_i. \tag{7}$$

According to [5], the mixture has no equation of state and application of Eq. (1) to determine the sound velocity of the mixture requires the concretization of its properties. The propagation velocity of a small perturbation (the sound) in any medium is determined by equation $C = \dfrac{\Delta L}{\Delta t}$, where ΔL is the distance passed by the perturbation in time Δt. We will start from this definition of sound velocity.

Let us consider the equilibrated mixture, each component of which is characterized by the following quantities: ρ_{i0} = const, $u_{i0} = 0$, $P_{i0} = P_0$ = const, S_{i0} = const, η_{i0} = const, α_{i0} = const, $F_{i0} = 0$, and C_{i0} = const. Let us formulate the set of simplifying hypotheses, in the context of which we derive various dependences of sound velocity in the mixture on the sound velocities of components:

(i) The sound is a small perturbation: $f = f_0 + \delta f$, $f_i = f_{i0} + \delta f_i$.

(ii) The sound is reversible: $\delta S = 0$, $S = S_0, \delta S_i = 0, S_i = S_{i0}$.

(iii) The relaxation time of the pressures equals zero: $\delta P_i = \delta P$.

(iv) All $\delta\alpha_i$ and $\delta\eta_i$ (i = 1, 2, ..., N) have the same sign.

The consequence of the fourth condition and Eqs. (7) has the form

$$\delta\alpha_i = 0,\quad \alpha_i = \alpha_{i0},\quad \delta\eta_i = 0,\quad \eta_i = \eta_{i0}. \tag{8}$$

To derive the C_i-dependences of C, let us consider Eq. (4). After its linearization allowing for (8), we derive the δV-dependence of δV_i

$$\delta V = \sum_{i=1}^{N}\eta_i\delta V_i.$$

Let us divide the left side of this equation by δP and the right side by $\delta P_i = \delta P$, and pass to the limit at $\delta P \to 0$, $\delta V \to 0$, and $\delta V_i \to 0$. As a result, according to (1), we derive the equation

$$\frac{1}{\rho^2C^2} = \sum_{i=1}^{N}\frac{\eta_i}{\rho_i^2C_i^2}. \tag{9}$$

Let us express η_i from third Eq. (7) and substitute it into (9). After the reduction of common multipliers, we derive Eq. (2).

In the context of the hypotheses formulated above, let us take Eq. (5) instead of Eq. (4). After its linearization, allowing for (8), we derive

$$\delta\rho = \sum_{i=1}^{N}\alpha_i\delta\rho_i.$$

Dividing this equation by $\delta P = \delta P_i$ and passing to the limit at $\delta P \to 0$, $\delta\rho \to 0, \delta\rho_i \to 0$, we derive the equation; using (1), it is transformed to the form

$$\frac{1}{C^2} = \sum_{i=1}^{N}\frac{\alpha_i}{C_i^2}. \tag{10}$$

When deriving Eq. (10), we used Eq. (5), and when deriving Eq. (2), we used Eq. (4). Although Eqs. (4) and (5) are identically exact, the consequences from them are different for the same simplifying hypotheses.

Let us consider the third method of deriving dependence $C = C(C_i, \alpha_i)$ in the context of the model of the multicomponent medium [5]. In the case of an ideal medium without thermal conductivity and chemical reactions, the conservation laws of the mass and pulse of the ith component in the one-dimensional case have the form

$$\frac{\partial}{\partial t}(\alpha_i\rho_i) + \frac{\partial}{\partial x}(\alpha_i\rho_iu_i) = 0, \tag{11}$$

$$\frac{\partial}{\partial t}(\alpha_i\rho_iu_i) + \frac{\partial}{\partial x}(\alpha_i\rho_iu_i^2) + \frac{\partial}{\partial x}(\alpha_i(P_i + F_i)) - R_i = 0. \tag{12}$$

The relation of pressure P_i with density ρ_i in the isentropic flow is established by the equation of state

$$P_i = P_i(\rho_i, S_i). \tag{13}$$

The dependences between the derivatives of density and pressure follow from (13) and (1):

$$\frac{\partial\rho_i}{\partial t} = \frac{1}{C_i^2}\frac{\partial P_i}{\partial t},\quad \frac{\partial\rho_i}{\partial x} = \frac{1}{C_i^2}\frac{\partial P_i}{\partial x}. \tag{14}$$

The conservation laws of the mass and pulse for the mixture of ideal media without thermal conductivity in the one-dimensional case are as follows:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{15}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial x}(P + F) = 0. \tag{16}$$

Let us restrict ourselves to the consideration of isentropic flows, in which

$$S_i = \text{const},\quad \frac{\partial S_i}{\partial t} = 0,\quad \frac{\partial S_i}{\partial x} = 0,$$

$$S = \text{const},\quad \frac{\partial S}{\partial t} = 0,\quad \frac{\partial S}{\partial x} = 0.$$

Let us pass to small perturbations in (11), (12) and (15), (16). It follows from [5] that functions F_i, F, R_i are the following:

$$\delta F_i = 0,\quad \delta F = 0,\quad \delta R_i = 0.$$

As a result, we derive four equations, which can be written in a characteristic form after the substitution of

Table

Substance	ρ_{0k}, g/cm^3	C_{0k}, km/s	γ	n	ρ_0, g/cm^3	E_0, kJ/g	C_0, km/s
W	19.35	4.051	2.67	3.6	19.2	0.07650	4.036
$C_{22}H_{46}$	0.930	3.357	1.667	3.5	0.91	0.364444	3.328

derivatives of ρ_i by derivatives of P_i using (14) and accounting for the requirements of four simplifying hypotheses. For the ith component and the mixture along the characteristics

$$\left(\frac{dx}{dt}\right)_i = \pm C_i, \quad \frac{dx}{dt} = \pm C$$

equations are valid:

$$\frac{\partial \delta P_i}{\partial t} \pm \rho_i C_i \frac{\partial \delta u_i}{\partial t} = 0, \quad \frac{\partial \delta P}{\partial t} \pm \rho C \frac{\partial \delta u}{\partial t} = 0. \tag{17}$$

After integration of Eqs. (17), in the traveling wave with the conservation of Z_i and Z invariants, we derive the relation of δu_i with δP_i and δu with δP:

$$\frac{\delta u_i}{\delta P_i} = \frac{1}{\rho_i C_i}, \quad \frac{\delta u}{\delta P} = \frac{1}{\rho C}. \tag{18}$$

Let us now use Eq. (6). After linearization allowing for (7), it has the form

$$\delta u = \sum_{i=1}^{N} \eta_i \delta u_i. \tag{19}$$

After dividing Eq. (19) by $\delta P_i = \delta P$ and substituting relations (18), we derive the equation

$$\frac{1}{\rho C} = \sum_{i=1}^{N} \frac{\eta_i}{\rho_i C_i}. \tag{20}$$

Expressing η_i from (7) and substituting it into (20), we derive the relation of C with C_i in the form

$$\frac{1}{C} = \sum_{i=1}^{N} \frac{\alpha_i}{C_i}. \tag{21}$$

The result can be formulated as follows. In the context of the same simplifying hypotheses, when selecting one of three Eqs. (4)–(6) identically rigorous in the theory of multicomponent media, we derive various C_i-dependences of C (2), (10) and (21).

To reveal the arguments in favor of one of the three dependences $C = C(C_i, \alpha_i)$, let us consider the mixture in the form of the set of planar layers, or components. Each ith layer has mass Δm_i, while the mixture as a whole has mass $\Delta m = \sum_{i=1}^{N} \Delta m_i$. When a planar shock wave is propagated along the layered system, it passes each ith layer with velocity W_i in time

$$\Delta t_i = \frac{\Delta m_i}{W_i}. \tag{22}$$

The shock wave will pass from one boundary of the mixture to another one in time

$$\Delta t = \sum_{i=1}^{N} \Delta t_i. \tag{23}$$

Let us determine the average velocity of the shock wave in the mixture by the equation

$$W = \frac{\Delta m}{\Delta t}. \tag{24}$$

Substituting Δt_i from (22) and Δt from (24) into (23), we derive the equation

$$\frac{\Delta m}{W} = \sum_{i=1}^{N} \frac{\Delta m_i}{W_i}. \tag{25}$$

The ratio of Δm_i to Δm is the weight concentration η_i. It is known from the theory of shock waves that

$$W = \rho(D - u), \quad W_i = \rho_i(D_i - u_i). \tag{26}$$

The sound perturbation is infinitesimally weak. For the infinitesimally weak shock wave

$$D = u + C, \quad D_i = u_i + C_i. \tag{27}$$

It follows from (25)–(27) that for the sound wave

$$\frac{1}{\rho C} = \sum_{i=1}^{N} \frac{\eta_i}{\rho_i C_i}. \tag{28}$$

Expressing η_i from (7), substituting η_i into (28), and reducing the common multipliers, we derive Eq. (21).

As one more argument, we computed the propagation of the shock wave along the layered system of planar layers of tungsten and paraffin using the VOLNA software [6]. Equations of state were taken in the form

$$P = (\gamma - 1)\rho E + \frac{\rho_{0k} C_{0k}^2}{n}\left(\frac{n-\gamma}{n-1}\delta^n + \frac{(\gamma-1)n}{n-1}\delta - \gamma\right).$$

The table represents the parameters of the equation of state and the initial characteristics of tungsten and paraffin at $P_0 = 10^{-4}$ GPa and $T_0 = 293$ K.

Computations for various α_W were performed for convergence by the number of the pairs of layers and by the amplitude of the initial perturbation tending to zero. The results of computations coincide with the computation of C by formula (21) accurate to six digits.

It follows from the aforesaid that the sound velocity in the multicomponent mixture is most exactly described by formula (21).

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